

HARDY-SOBOLEV EQUATIONS WITH ASYMPTOTICALLY VANISHING SINGULARITY: BLOW-UP ANALYSIS FOR THE MINIMAL ENERGY

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ABSTRACT. We study the asymptotic behavior of a sequence of positive solutions $(u_\epsilon)_{\epsilon>0}$ as $\epsilon \rightarrow 0$ to the family of equations

$$\begin{cases} \Delta u_\epsilon + a(x)u_\epsilon = \frac{u_\epsilon^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

where $(s_\epsilon)_{\epsilon>0}$ is a sequence of positive real numbers such that $\lim_{\epsilon \rightarrow 0} s_\epsilon = 0$, $2^*(s_\epsilon) := \frac{2(n-s_\epsilon)}{n-2}$ and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain such that $0 \in \partial\Omega$. When the sequence $(u_\epsilon)_{\epsilon>0}$ is uniformly bounded in L^∞ , then upto a subsequence it converges strongly to a minimizing solution of the stationary Schrödinger equation with critical growth. In case the sequence blows up, we obtain strong pointwise control on the blow up sequence, and then using the Pohozaev identity localize the point of singularity, which in this case can at most be one, and derive precise blow up rates. In particular when $n = 3$ or $a \equiv 0$ then blow up can occur only at an interior point of Ω or the point $0 \in \partial\Omega$.

1. INTRODUCTION

Let Ω be a bounded smooth oriented domain of \mathbb{R}^n , $n \geq 3$, such that $0 \in \partial\Omega$. We define the Sobolev space $H_{1,0}^2(\Omega)$ as the completion of the space $C_c^\infty(\Omega)$, the space of compactly supported smooth functions in Ω , with respect to the norm $u \mapsto \|u\|_{H_{1,0}^2(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$. We let $2^* := \frac{2n}{n-2}$ be the critical Sobolev exponent for the embedding $H_{1,0}^2(\Omega) \hookrightarrow L^p(\Omega)$. Namely, the embedding is defined and continuous for $1 \leq p \leq 2^*$, and it is compact iff $1 \leq p < 2^*$. Let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω , that is there exists $A_0 > 0$ such that $\int_\Omega (|\nabla \varphi|^2 + a\varphi^2) dx \geq A_0 \|u\|_{H_{1,0}^2(\Omega)}^2$ for all $\varphi \in H_{1,0}^2(\Omega)$. Solutions $u \in C^2(\overline{\Omega})$ to the problem

$$\begin{cases} \Delta u + a(x)u = u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(often referred to as "Brezis-Nirenberg problem") are critical points of the functional

$$u \mapsto J(u) := \frac{\int_\Omega (|\nabla u|^2 + au^2) dx}{\left(\int_\Omega |u|^{2^*} dx \right)^{2/2^*}}.$$

Date: February 8th, 2017.

2010 Mathematics Subject Classification. 35J60, 35B40.

This work is part of the PhD thesis of the author, funded by "Fédération Charles Hermite" (FR3198 du CNRS) and "Région Lorraine". The author acknowledges these two institutions for their supports.

Here, $\Delta := -\operatorname{div}(\nabla) = -\sum_i \partial_{ii}$ is the Laplacian with minus sign convention. A natural way to obtain such critical points is to find minimizers to this functional, that is to prove that

$$\mu_a(\Omega) = \inf_{u \in H_{1,0}^2(\Omega) \setminus \{0\}} J(u)$$

is achieved. There is a huge and extensive litterature on this problem, starting with the pioneering article of Brezis-Nirenberg [4] in which the authors completely solved the question of existence of minimizers for $\mu_a(\Omega)$ when $a \equiv \text{constant}$ and $n \geq 4$ for any domain, and $n = 3$ for a ball. Their analysis took inspiration from the contributions of Aubin [2] in the resolution of the Yamabe problem. The case when a is arbitrary and $n = 3$ was solved by Druet [6] using blowup analysis.

In [11], Ghoussoub-Kang suggested an alternative approach by adding a singularity in the equation as follows. For any $s \in [0, 2)$, we define

$$2^*(s) := \frac{2(n-s)}{n-2}$$

so that $2^* = 2^*(0)$. Consider the weak solutions $u \in H_{1,0}^2(\Omega) \setminus \{0\}$ to the problem

$$\begin{cases} \Delta u + a(x)u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note here that $0 \in \partial\Omega$ is a boundary point. Such solutions can be achieved as minimizers for the problem

$$(1) \quad \mu_{s,a}(\Omega) = \inf_{u \in H_{1,0}^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + au^2) dx}{\left(\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{2/2^*(s)}} \quad \text{for } s \in (0, 2)$$

Consider a sequence of positive real numbers $(s_{\epsilon})_{\epsilon>0}$ such that $\lim_{\epsilon \rightarrow 0} s_{\epsilon} = 0$. We let $(u_{\epsilon})_{\epsilon>0} \in C^2(\overline{\Omega} \setminus \{0\}) \cap C^1(\overline{\Omega})$ such that

$$(2) \quad \begin{cases} \Delta u_{\epsilon} + au_{\epsilon} = \frac{u_{\epsilon}^{2^*(s_{\epsilon})-1}}{|x|^{s_{\epsilon}}} & \text{in } \Omega, \\ u_{\epsilon} > 0 & \text{in } \Omega, \\ u_{\epsilon} = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, we assume that the (u_{ϵ}) 's are of minimal energy type in the sense that

$$(3) \quad \frac{\int_{\Omega} (|\nabla u_{\epsilon}|^2 + au_{\epsilon}^2) dx}{\left(\int_{\Omega} \frac{|u_{\epsilon}|^{2^*(s_{\epsilon})}}{|x|^{s_{\epsilon}}} dx \right)^{2/2^*(s_{\epsilon})}} = \mu_{s_{\epsilon},a}(\Omega) + o(1) \leq \frac{1}{K(n,0)} + o(1)$$

as $\epsilon \rightarrow 0$, where $K(n,0) > 0$ is the best constant in the Sobolev embedding defined in (5). Indeed, it follows from Ghoussoub-Robert [9, 10] that such a family $(u_{\epsilon})_{\epsilon}$ exists if the the mean curvature of $\partial\Omega$ at 0 is negative.

In this paper we are interested in studying the asymptotic behavior of the sequence $(u_{\epsilon})_{\epsilon>0}$ as $\epsilon \rightarrow 0$. As proved in Proposition 2.2, if the weak limit u_0 of $(u_{\epsilon})_{\epsilon}$ in $H_{1,0}^2(\Omega)$ is nontrivial, then the convergence is indeed strong and u_0 is a minimizer of $\mu_a(\Omega)$. We completely deal with the case $u_0 \equiv 0$, which is more delicate, in which

blow-up necessarily occurs. In the spirit of the C^0 –theory of Druet-Hebey-Robert [7], our first result is the following:

Theorem 1. *Let Ω be a bounded smooth oriented domain of \mathbb{R}^n , $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω . Let $(s_\epsilon)_{\epsilon>0} \in (0, 2)$ be a sequence such that $\lim_{\epsilon \rightarrow 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (2) and (3), is a blowup sequence, i.e*

$$u_\epsilon \rightharpoonup 0 \quad \text{weakly in } H_{1,0}^2(\Omega) \quad \text{as } \epsilon \rightarrow 0$$

Then, there exists $C > 0$ such that for all $\epsilon > 0$

$$u_\epsilon(x) \leq C \left(\frac{\mu_\epsilon}{\mu_\epsilon^2 + |x - x_\epsilon|^2} \right)^{\frac{n-2}{2}} \quad \text{for all } x \in \Omega$$

where $\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x \in \Omega} u_\epsilon(x)$.

With this optimal pointwise control, we to obtain more informations on the localization of the blowup point $x_0 := \lim_{\epsilon \rightarrow 0} x_\epsilon$ and the blowup parameter $(\mu_\epsilon)_\epsilon$. We let $G : \overline{\Omega} \times \overline{\Omega} \setminus \{(x, x) : x \in \overline{\Omega}\} \rightarrow \mathbb{R}$ be the Green's function of the coercive operator $\Delta + a$ in Ω with Dirichlet boundary conditions. For any $x \in \Omega$ we write G_x as:

$$G_x(y) = \frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}} + g_x(y) \quad \text{for } y \in \Omega \setminus \{x\}$$

where ω_{n-1} is the area of the $(n-1)$ - sphere. In dimension $n = 3$ or when $a \equiv 0$, one has that $g_x \in C^2(\overline{\Omega} \setminus \{x\}) \cap C^{0,\theta}(\Omega)$ for some $0 < \theta < 1$, and $g_x(x)$ is defined for all $x \in \Omega$ and is called the mass of the operator $\Delta + a$.

Theorem 2. *Let Ω be a bounded smooth oriented domain of \mathbb{R}^n , $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω . Let $(s_\epsilon)_{\epsilon>0} \in (0, 2)$ be a sequence such that $\lim_{\epsilon \rightarrow 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (2) and (3), is a blowup sequence, i.e*

$$u_\epsilon \rightharpoonup 0 \quad \text{weakly in } H_{1,0}^2(\Omega) \quad \text{as } \epsilon \rightarrow 0$$

We let $(\mu_\epsilon)_\epsilon \in (0, +\infty)$ and $(x_\epsilon)_\epsilon \in \Omega$ be such that $\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x \in \Omega} u_\epsilon(x)$.

We define $x_0 := \lim_{\epsilon \rightarrow 0} x_\epsilon$ and we assume that

$$x_0 \in \Omega \text{ is an interior point.}$$

Then

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{s_\epsilon}{\mu_\epsilon^2} &= 2^* K(n, 0)^{\frac{2^*}{2^*-2}} d_n a(x_0) & \text{for } n \geq 5 \\ \lim_{\epsilon \rightarrow 0} \frac{s_\epsilon}{\mu_\epsilon^2 \log(1/\mu_\epsilon)} &= 256 \omega_3 K(4, 0)^2 a(x_0) & \text{for } n = 4 \\ \lim_{\epsilon \rightarrow 0} \frac{s_\epsilon}{\mu_\epsilon^{\frac{n-2}{2}}} &= -n b_n^2 K(n, 0)^{n/2} g_{x_0}(x_0) & \text{for } n = 3 \text{ or } a \equiv 0. \end{aligned}$$

where $g_{x_0}(x_0)$ the mass at the point $x_0 \in \Omega$ for the operator $\Delta + a$, and
(4)

$$d_n = \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-(n-2)} dx \text{ for } n \geq 5; \quad b_n = \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-\frac{n+2}{2}} dx$$

and ω_3 is the area of the 3-sphere.

When $x_0 \in \partial\Omega$ is a boundary point, we get similar estimates:

Theorem 3. *Let Ω be a bounded smooth oriented domain of \mathbb{R}^n , $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\bar{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω . Let $(s_\epsilon)_{\epsilon>0} \in (0, 2)$ be a sequence such that $\lim_{\epsilon \rightarrow 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (2) and (3), is a blowup sequence, i.e*

$$u_\epsilon \rightharpoonup 0 \quad \text{weakly in } H_{1,0}^2(\Omega) \quad \text{as } \epsilon \rightarrow 0$$

We let $(\mu_\epsilon)_\epsilon \in (0, +\infty)$ and $(x_\epsilon)_\epsilon \in \Omega$ be such that $\mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon) = \max_{x \in \Omega} u_\epsilon(x)$. Assume that

$$\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in \partial\Omega.$$

Then

(1) If $n = 3$ or $a \equiv 0$, then as $\epsilon \rightarrow 0$

$$\lim_{\epsilon \rightarrow 0} \frac{s_\epsilon d(x_\epsilon, \partial\Omega)^{n-2}}{\mu_\epsilon^{n-2}} = \frac{n^{n-1}(n-2)^{n-1}K(n, 0)^{n/2}\omega_{n-1}}{2^{n-2}}.$$

Moreover, $d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_\epsilon|$ as $\epsilon \rightarrow 0$. In particular $x_0 = 0$.

(2) If $n = 4$. Then as $\epsilon \rightarrow 0$

$$\frac{s_\epsilon}{4} (K(4, 0)^{-2} + o(1)) - \left(\frac{\mu_\epsilon}{d(x_\epsilon, \partial\Omega)}\right)^2 (32\omega_3 + o(1)) = \mu_\epsilon^2 \log\left(\frac{d(x_\epsilon, \partial\Omega)}{\mu_\epsilon}\right) [64\omega_3 a(x_0) + o(1)]$$

(3) If $n \geq 5$. Then as $\epsilon \rightarrow 0$

$$\frac{s_\epsilon(n-2)}{2n} \left(K(n, 0)^{-n/2} + o(1)\right) - \left(\frac{\mu_\epsilon}{d(x_\epsilon, \partial\Omega)}\right)^{n-2} \left(\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}} + o(1)\right) = \mu_\epsilon^2 [d_n a(x_0) + o(1)]$$

where d_n is as in (4).

Theorem 3 is a particular case of Theorem 7 proved in Section 7.

The main difficulty in our analysis is due to the natural singularity at $0 \in \partial\Omega$. Indeed, there is a balance between two facts. First, since $s_\epsilon > 0$, this singularity exists and has an influence on the analysis, and in particular on the Pohozaev identity (see the statement of Theorem 2). But, second, since $s_\epsilon \rightarrow 0$, the singularity should cancel, at least asymptotically. In this perspective, our results are twofolds.

Theorem 1 asserts that the pointwise control is the same as the control of the classical problem with $s_\epsilon = 0$: however, to prove this result, we need to perform a very delicate analysis of the blowup with the perturbation $s_\epsilon > 0$, even for the initial steps that are usually standard when $s_\epsilon = 0$ (these are Sections 3 and 4).

The influence and the role of $s_\epsilon > 0$ is much more striking in Theorems 2 and 3. Compared to the case $s_\epsilon = 0$, the Pohozaev identity (see Section 6) enjoys an additional term involving s_ϵ that is present in the statement of Theorems 2 and 3.

Heuristically, this is due to the fact that the limiting equation $\Delta u = |x|^{-s} u^{2^*(s)-1}$ is not invariant under the action of the translations when $s > 0$.

Some classical references for the blow-up analysis of nonlinear critical elliptic pdes are Rey [18], Adimurthi- Pacella-Yadava [1], Han [12], Hebey-Vaugon [14] and Khuri-Marques-Schoen [16]. In Mazumdar [17] the usefulness of blow-up analysis techniques were illustrated by proving the existence of solution to critical growth polyharmonic problems on manifolds. The analysis of the 3 dimensional problem by Druet [6] and the monograph [7] by Druet-Hebey-Robert were important sources of inspiration.

This paper is organized as follows. In Section 2 we recall general facts on Hardy-Sobolev inequalities and prove few useful general and classical statements. Section 3 is devoted to the proof of convergence to a ground state up to rescaling. In Section 4, we perform a delicate blow-up analysis to get a first pointwise control on u_ϵ . The optimal control of Theorem 1 is proved in Section 5. With the pointwise control of Theorem 1, we are able to estimate the maximum of the u_ϵ 's when the blowup point is in the interior of the domain (Section 6) or on the boundary (Section 7).

Acknowledgements. I would like to express my deep gratitude to Professor Frédéric Robert and Professor Dong Ye, my thesis supervisors, for their patient guidance, enthusiastic encouragement and useful critiques of this work.

2. HARDY-SOBOLEV INEQUALITY AND THE CASE OF A NONZERO WEAK LIMIT

The space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is defined as the completion of the space $C_c^\infty(\mathbb{R}^n)$, the space of compactly supported smooth functions in \mathbb{R}^n , with respect to the norm $\|u\|_{\mathcal{D}^{1,2}} = \|\nabla u\|_{L^2(\mathbb{R}^n)}$. The embedding $\mathcal{D}^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is continuous, and we denote the best constant of this embedding by $K(n, 0)$ which can be characterised as

$$(5) \quad \frac{1}{K(n, 0)} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2^*} dx \right)^{2/2^*}}$$

Interpolating the Sobolev inequality and the Hardy inequality

$$(6) \quad \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} dx \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx \text{ for } u \in \mathcal{D}^{1,2}(\mathbb{R}^n),$$

we get the so-called "Hardy-Sobolev inequality" (see [11] and the references therein): there exists a constant $K(n, s) > 0$ such that

$$(7) \quad \frac{1}{K(n, s)} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{2/2^*(s)}}$$

As one checks, $\lim_{s \rightarrow 0} K(n, s) = K(n, 0)$. For a domain $\Omega \subset \mathbb{R}^n$, we also have:

Proposition 2.1. $\lim_{s \rightarrow 0} \mu_{s,a}(\Omega) = \mu_a(\Omega)$.

Proof. Let $u \in H_{1,0}^2(\Omega) \setminus \{0\}$. Hölder and Hardy inequalities yield

$$\left(\int_{\Omega} \frac{|u(x)|^{2^*(s)}}{|x|^s} dx \right)^{2/2^*(s)} \leq \left(\frac{2}{n-2} \right)^{2s/2^*(s)} \left(\int_{\Omega} |\nabla u(x)|^2 dx \right)^{s/2^*(s)} \left(\int_{\Omega} |u(x)|^{2^*} dx \right)^{\frac{2-s}{2^*(s)}}$$

then the Sobolev inequality gives that for all $u \in H_{1,0}^2(\Omega) \setminus \{0\}$ one has

$$\frac{\int_{\Omega} (|\nabla u|^2 + au^2) dx}{\left(\int_{\Omega} |u|^{2^*} dx \right)^{2/2^*}} \leq \frac{\int_{\Omega} (|\nabla u|^2 + au^2) dx}{\left(\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{2/2^*(s)}} \left(\frac{1}{K(n,0)^{1/2^*}} \frac{2}{n-2} \right)^{s \frac{n-2}{n-s}}$$

So $\mu_a(\Omega) \leq \mu_{s,a}(\Omega) \left(\frac{1}{K(n,0)^{1/2^*}} \frac{2}{n-2} \right)^{s \frac{n-2}{n-s}}$. Passing to limits as $s \rightarrow 0$, one obtains that $\mu_a(\Omega) \leq \liminf_{s \rightarrow 0} \mu_{s,a}(\Omega)$. Let $u \in H_{1,0}^2(\Omega) \setminus \{0\}$. By Fatou's lemma one has

$$\begin{aligned} \int_{\Omega} |u(x)|^{2^*} dx &\leq \liminf_{s \rightarrow 0} \int_{\Omega} \frac{|u(x)|^{2^*(s)}}{|x|^s} dx \leq \liminf_{s \rightarrow 0} \left(\frac{1}{\mu_{s,a}(\Omega)} \int_{\Omega} (|\nabla u|^2 + au^2) dx \right)^{2^*(s)/2}, \\ &\left(\int_{\Omega} |u(x)|^{2^*} dx \right)^{2/2^*} \leq \liminf_{s \rightarrow 0} \frac{1}{\mu_{s,a}(\Omega)} \int_{\Omega} (|\nabla u|^2 + au^2) dx \end{aligned}$$

Therefore $\limsup_{s \rightarrow 0} \mu_{s,a}(\Omega) \leq \mu_a(\Omega)$, hence $\lim_{s \rightarrow 0} \mu_{s,a}(\Omega) = \mu_a(\Omega)$. This proves Proposition 2.1. \square

The following proposition is standard:

Proposition 2.2. *Let Ω be a bounded smooth oriented domain of \mathbb{R}^n , $n \geq 3$, such that $0 \in \partial\Omega$. Let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω . Let $(u_{\epsilon})_{\epsilon > 0} \in C^2(\overline{\Omega} \setminus \{0\}) \cap C^1(\overline{\Omega})$ be as in (2) and (3). Then there exists $u_0 \in H_{1,0}^2(\Omega)$ such that, up to extraction, $u_{\epsilon} \rightharpoonup u_0$ weakly in $H_{1,0}^2(\Omega)$ as $\epsilon \rightarrow 0$. Indeed, $u_0 \in C^2(\overline{\Omega} \setminus \{0\}) \cap C^1(\overline{\Omega})$ is a solution to*

$$\begin{cases} \Delta u_0 + au_0 = u_0^{2^*-1} & \text{in } \Omega \\ u_0 \geq 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

If $u_0 \neq 0$, then $u_0 > 0$ in Ω and $\lim_{\epsilon \rightarrow 0} u_{\epsilon} = u_0$ in $C^1(\overline{\Omega})$. Moreover, $\mu_a(\Omega)$ is achieved by u_0 .

3. PRELIMINARY BLOW-UP ANALYSIS

We define $\mathbb{R}_-^n = \{x \in \mathbb{R}^n : x_1 < 0\}$ where x_1 is the first coordinate of a generic point in \mathbb{R}^n . This space will be the limit space in certain cases after blowup. We describe a parametrisation around a point of the boundary $\partial\Omega$. Let $p \in \partial\Omega$. Then there exists U, V open in \mathbb{R}^n and a smooth diffeomorphism $\mathcal{T} : U \rightarrow V$ such that

upto a rotation of coordinates if necessary

$$(8) \quad \left\{ \begin{array}{l} \bullet \quad 0 \in U \text{ and } p \in V \\ \bullet \quad \mathcal{T}(0) = p \\ \bullet \quad \mathcal{T}(U \cap \{x_1 < 0\}) = V \cap \Omega \\ \bullet \quad \mathcal{T}(U \cap \{x_1 = 0\}) = V \cap \partial\Omega \\ \bullet \quad D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}. \text{ Here } D_x\mathcal{T} \text{ denotes the differential of } \mathcal{T} \text{ at the point } x \\ \quad \text{and } \mathbb{I}_{\mathbb{R}^n} \text{ is the identity map on } \mathbb{R}^n. \\ \bullet \quad D_0\mathcal{T}(e_1) = \nu_p \text{ where } \nu_p \text{ denotes the outer unit normal vector to} \\ \quad \partial\Omega \text{ at the point } p. \\ \bullet \quad \{D_0\mathcal{T}(e_2), \dots, D_0\mathcal{T}(e_n)\} \text{ forms an orthonormal basis of } T_p\partial\Omega. \end{array} \right.$$

We start with a scaling lemma which we shall employ many times in our analysis.

Lemma 1. *Let Ω be a bounded smooth oriented domain of \mathbb{R}^n , $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω . Let $(s_\epsilon)_{\epsilon>0} \in (0, 2)$ be a sequence such that $\lim_{\epsilon \rightarrow 0} s_\epsilon = 0$. Consider the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (2) and (3). Let $(y_\epsilon)_\epsilon \in \Omega$, and let $(\nu_\epsilon)_\epsilon$ and $(\beta_\epsilon)_\epsilon$ be sequences of positive real numbers defined by*

$$(9) \quad \nu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(y_\epsilon) \quad \beta_\epsilon := |y_\epsilon|^{s_\epsilon/2} \nu_\epsilon^{\frac{2-s_\epsilon}{2}} \quad \text{for } \epsilon > 0$$

Suppose that $\lim_{\epsilon \rightarrow 0} \nu_\epsilon = 0$ which then implies that $\lim_{\epsilon \rightarrow 0} \beta_\epsilon = 0$. Assume that there exists $C_1 > 0$ such that for any given $R > 0$ one has for $\epsilon > 0$ small

$$(10) \quad u_\epsilon(x) \leq C_1 u_\epsilon(y_\epsilon) \quad \text{for all } x \in B_{y_\epsilon}(R\nu_\epsilon)$$

Then $\nu_\epsilon = o(|y_\epsilon|)$ as $\epsilon \rightarrow 0$. Along with the above assumption also suppose that there exists $C_2 > 0$ such that for any given $R > 0$ one has for $\epsilon > 0$ small

$$(11) \quad u_\epsilon(x) \leq C_2 u_\epsilon(y_\epsilon) \quad \text{for all } x \in B_{y_\epsilon}(R\beta_\epsilon)$$

Then $\beta_\epsilon = o(d(y_\epsilon, \partial\Omega))$ as $\epsilon \rightarrow 0$. For $\epsilon > 0$ we then rescale and define

$$(12) \quad w_\epsilon(x) := \frac{u_\epsilon(y_\epsilon + \beta_\epsilon x)}{u_\epsilon(y_\epsilon)} \quad \text{for } x \in \frac{\Omega - y_\epsilon}{\beta_\epsilon}$$

Then there exists $w \in C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,2}(\mathbb{R}^n)$ such that $w > 0$ and for any $\eta \in C_c^\infty(\mathbb{R}^n)$

$$\eta w_\epsilon \rightharpoonup \eta w \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^n) \quad \text{as } \epsilon \rightarrow 0$$

Further, $\lim_{\epsilon \rightarrow 0} w_\epsilon = w$ in $C_{loc}^1(\mathbb{R}^n)$ and w satisfies the equation

$$\begin{cases} \Delta w = w^{2^*-1} & \text{in } \mathbb{R}^n \\ w \geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Proof. The proof is completed in the following steps.

Step 1: We claim that

$$(13) \quad \lim_{\epsilon \rightarrow 0} \frac{|y_\epsilon|}{\nu_\epsilon} = +\infty$$

We prove our claim. Suppose on the contrary that $\frac{|y_\epsilon|}{\nu_\epsilon} = O(1)$ as $\epsilon \rightarrow 0$. Then $\lim_{\epsilon \rightarrow +\infty} |y_\epsilon| = 0$. Let $\mathcal{T} : U \rightarrow V$ be a parametrisation of the boundary as in (8) at the point $p = 0$. For all $\epsilon > 0$, we let

$$\tilde{w}_\epsilon(x) = \frac{u_\epsilon \circ \mathcal{T}(\nu_\epsilon x)}{u_\epsilon(y_\epsilon)} \quad \text{for } x \in \frac{U}{\nu_\epsilon} \cap \{x_1 \leq 0\}$$

Step 1.1: For any $\eta \in C_c^\infty(\mathbb{R}^n)$, one has that $\eta \tilde{w}_\epsilon \in \mathcal{D}^{1,2}(\mathbb{R}_-^n)$ for $\epsilon > 0$ sufficiently small. We claim that there exists $\tilde{w}_\eta \in \mathcal{D}^{1,2}(\mathbb{R}_-^n)$ such that upto a subsequence

$$\begin{cases} \eta \tilde{w}_\epsilon \rightharpoonup \tilde{w}_\eta & \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}_-^n) \text{ as } \epsilon \rightarrow 0 \\ \eta \tilde{w}_\epsilon(x) \rightarrow \tilde{w}_\eta(x) & \text{a.e. in } \mathbb{R}_-^n \text{ as } \epsilon \rightarrow 0 \end{cases}$$

We prove the claim. Let $x \in \mathbb{R}_-^n$, then

$$\nabla(\eta \tilde{w}_\epsilon)(x) = \tilde{w}_\epsilon(x) \nabla \eta(x) + \frac{\nu_\epsilon}{u_\epsilon(y_\epsilon)} \eta(x) D_{(\nu_\epsilon x)} \mathcal{T}[\nabla u_\epsilon(\mathcal{T}(\nu_\epsilon x))]$$

Now for any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $a, b > 0$, $(a+b)^2 \leq C(\theta)a^2 + (1+\theta)b^2$. With this inequality we then obtain

$$\int_{\mathbb{R}_-^n} |\nabla(\eta \tilde{w}_\epsilon)|^2 dx \leq C(\theta) \int_{\mathbb{R}_-^n} |\nabla \eta|^2 \tilde{w}_\epsilon^2 dx + (1+\theta) \frac{\nu_\epsilon^2}{u_\epsilon^2(y_\epsilon)} \int_{\mathbb{R}_-^n} \eta^2 |D_{(\nu_\epsilon x)} \mathcal{T}[\nabla u_\epsilon(\mathcal{T}(\nu_\epsilon x))]|^2 dx$$

Since $D_0 \mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$ we have as $\epsilon \rightarrow 0$

$$\int_{\mathbb{R}_-^n} |\nabla(\eta \tilde{w}_\epsilon)|^2 dx \leq C(\theta) \int_{\mathbb{R}_-^n} |\nabla \eta|^2 \tilde{w}_\epsilon^2 dx + (1+\theta)(1+O(\nu_\epsilon)) \frac{\nu_\epsilon^2}{u_\epsilon^2(y_\epsilon)} \int_{\mathbb{R}_-^n} \eta^2 |\nabla u_\epsilon(\mathcal{T}(\nu_\epsilon x))|^2 (1+o(1)) dx$$

With Hölder inequality and a change of variables this becomes

(14)

$$\int_{\mathbb{R}_-^n} |\nabla(\eta \tilde{w}_\epsilon)|^2 dx \leq C(\theta) \|\nabla \eta\|_{L^n}^2 \left(\int_{\Omega} u_\epsilon^{2^*} dx \right)^{\frac{n-2}{n}} + (1+\theta)(1+O(\nu_\epsilon)) \int_{\Omega} |\nabla u_\epsilon|^2 dx$$

Since $\|u_\epsilon\|_{H_{1,0}^2(\Omega)} = O(1)$ and $\nu_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, so for $\epsilon > 0$ small enough, $\|\eta \tilde{w}_\epsilon\|_{\mathcal{D}^{1,2}(\mathbb{R}_-^n)} \leq C_\eta$, where C_η is a constant depending on the function η . The claim then follows from the reflexivity of $\mathcal{D}^{1,2}(\mathbb{R}_-^n)$.

Step 1.2: Via a diagonal argument, we get that there exists $\tilde{w} \in \mathcal{D}^{1,2}(\mathbb{R}_-^n)$ such that for any $\eta \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{cases} \eta \tilde{w}_\epsilon \rightharpoonup \eta \tilde{w} & \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}_-^n) \text{ as } \epsilon \rightarrow 0 \\ \eta \tilde{w}_\epsilon(x) \rightarrow \eta \tilde{w}(x) & \text{a.e. in } \mathbb{R}_-^n \text{ as } \epsilon \rightarrow 0 \end{cases}$$

We claim that $\tilde{w} \in C^1(\overline{\mathbb{R}_-^n})$ and it satisfies weakly the equation

$$(15) \quad \begin{cases} \Delta \tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}_-^n \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

We prove the claim. For $i, j = 1, \dots, n$, we let $g_{ij} = (\partial_i \mathcal{T}, \partial_j \mathcal{T})$, the metric induced by the chart \mathcal{T} on the domain $U \cap \{x_1 < 0\}$ and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g . We let

$$\tilde{g}_\epsilon = g(\nu_\epsilon x)$$

From (2) it follows that for any $\epsilon > 0$ and $R > 0$, \tilde{w}_ϵ satisfies weakly the equation

$$(16) \quad \begin{aligned} \Delta \tilde{w}_\epsilon + \nu_\epsilon^2 (a \circ \mathcal{T}(\nu_\epsilon x)) \tilde{w}_\epsilon &= \frac{\tilde{w}_\epsilon^{2^*(s_\epsilon)-1}}{\left| \frac{\mathcal{T}(\nu_\epsilon x)}{\nu_\epsilon} \right|^{s_\epsilon}} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \tilde{w}_\epsilon &= 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{aligned}$$

From (10) and the properties of the boundary chart \mathcal{T} it follows that there exists $C_1 > 0$ such that for $\epsilon > 0$ small $0 \leq \tilde{w}_\epsilon(x) \leq C_1$ for all $x \in B_0(R) \cap \{x_1 \leq 0\}$, for $R > 0$ large. Then for any $p > 1$ there exists a constant $C_p > 0$ such that

$$\int_{B_0(R) \cap \{x_1 < 0\}} \left[\frac{(\tilde{w}_\epsilon)^{2^*(s_\epsilon)-1}}{\left| \frac{\mathcal{T}(\nu_\epsilon x)}{\nu_\epsilon} \right|^{s_\epsilon}} \right]^p dx \leq C_p \int_{B_0(R) \cap \{x_1 < 0\}} \frac{1}{|x|^{s_\epsilon p}} dx$$

So the right hand side of equation (16) is uniformly bounded in L^p for some $p > n$. From standard elliptic estimates it follows that the sequence $(\eta_R \tilde{w}_\epsilon)_{\epsilon > 0}$ is bounded in $C^{1,\alpha_0}(B_0(R) \cap \{x_1 \leq 0\})$ for some $\alpha_0 \in (0, 1)$. So by Arzela-Ascoli's theorem and a diagonal argument, we get that $\tilde{w} \in C_{loc}^{1,\alpha}(\mathbb{R}^n \cap \{x_1 \leq 0\})$ for $0 < \alpha < \alpha_0$, and that, up to a subsequence

$$\lim_{\epsilon \rightarrow 0} \tilde{w}_\epsilon = \tilde{w} \quad \text{in } C_{loc}^{1,\alpha}(\mathbb{R}^n \cap \{x_1 \leq 0\})$$

for $0 < \alpha < \alpha_0$. Passing to the limit in (16), we get (15). This proves our claim.

Step 1.3: Let $\tilde{y}_\epsilon \in U$ be such that $\mathcal{T}(\tilde{y}_\epsilon) = y_\epsilon$. From the properties (8) of the boundary chart \mathcal{T} , we get that $\frac{|\tilde{y}_\epsilon|}{\nu_\epsilon} = O\left(\frac{|y_\epsilon|}{\nu_\epsilon}\right)$. Then there exists $\tilde{y} \in \overline{\mathbb{R}^n_-}$ such that $\frac{\tilde{y}_\epsilon}{\nu_\epsilon} \rightarrow \tilde{y}$ as $\epsilon \rightarrow 0$. Therefore $\tilde{w}(\tilde{y}) = \lim_{\epsilon \rightarrow 0} \tilde{w}_\epsilon(\nu_\epsilon^{-1} \tilde{y}_\epsilon) = 1$. Therefore $\tilde{y} \in \mathbb{R}^n_-$, and then $\tilde{w} \in C^1(\overline{\mathbb{R}^n_-})$ is a nontrivial weak solution of the equation

$$\begin{cases} \Delta \tilde{w} = \tilde{w}^{2^*-1} & \text{in } \mathbb{R}^n \\ \tilde{w} = 0 & \text{on } \{x_1 = 0\} \end{cases}$$

which is impossible, see Struwe [20] (Chapter III, Theorem 1.3) and the Liouville theorem on half space. Hence (13) holds. This completes the proof of Step 1.

Step 2: Next, arguing similarly as in Step 1 and using (13), we get that

$$(17) \quad \lim_{\epsilon \rightarrow 0} \frac{d(y_\epsilon, \partial\Omega)}{\beta_\epsilon} = +\infty$$

We define w_ϵ as in (12). We fix $\eta \in C_c^\infty(\mathbb{R}^n)$. Then $\eta w_\epsilon \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ for $\epsilon > 0$ small. Arguing as in Step 1, for any $\theta > 0$, there exists $C(\theta) > 0$ such that

$$(18) \quad \begin{aligned} \int_{\mathbb{R}^n} |\nabla(\eta w_\epsilon)|^2 dx &\leq \left(\frac{\nu_\epsilon}{\beta_\epsilon}\right)^{n-2} C(\theta) \|\nabla \eta\|_{L^n}^2 \left(\int_{\mathbb{R}^n} u_\epsilon^{2^*} dx \right)^{\frac{n-2}{n}} \\ &\quad + (1+\theta) \left(\frac{\nu_\epsilon}{\beta_\epsilon}\right)^{n-2} \int_{\mathbb{R}^n} \left(\eta \left(\frac{x - y_\epsilon}{\beta_\epsilon} \right) \right)^2 |\nabla u_\epsilon|^2 dx. \end{aligned}$$

Arguing as in Step 1, $(\eta w_\epsilon)_\epsilon$ is uniformly bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, and there exists $w \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ such that upto a subsequence

$$(19) \quad \begin{cases} \eta w_\epsilon \rightharpoonup \eta w & \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^n) \text{ as } \epsilon \rightarrow 0 \\ \eta w_\epsilon(x) \rightarrow \eta w(x) & \text{a.e } x \text{ in } \mathbb{R}^n \text{ as } \epsilon \rightarrow 0 \end{cases}$$

Further $w \in C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,2}(\mathbb{R}^n)$, $w \geq 0$ and it satisfies weakly the equation $\Delta w = w^{2^*-1}$ in \mathbb{R}^n . Moreover $\lim_{\epsilon \rightarrow 0} w_\epsilon = w$ in $C_{loc}^1(\mathbb{R}^n)$, $w(0) = 1$ and $w > 0$. This ends Step 2 and proves Lemma 1. \square

We let (u_ϵ) be as in Theorem 1. We will say that *blowup occurs* whenever $u_\epsilon \rightharpoonup 0$ weakly in $H_{1,0}^2(\Omega)$ as $\epsilon \rightarrow 0$. We describe the behaviour of such a sequence of solutions (u_ϵ) . By regularity, for all ϵ , $u_\epsilon \in C^0(\overline{\Omega})$. We let $x_\epsilon \in \Omega$ and $\mu_\epsilon > 0$ be such that :

$$(20) \quad u_\epsilon(x_\epsilon) = \max_{\overline{\Omega}} u_\epsilon(x) \quad \text{and} \quad \mu_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(x_\epsilon)$$

The main result of this section is the following theorem:

Theorem 4. *Let Ω be a bounded smooth oriented domain of \mathbb{R}^n , $n \geq 3$, such that $0 \in \partial\Omega$, and let $a \in C^1(\overline{\Omega})$ be such that the operator $\Delta + a$ is coercive in Ω . Let $(s_\epsilon)_{\epsilon>0} \in (0, 2)$ be a sequence such that $\lim_{\epsilon \rightarrow 0} s_\epsilon = 0$. Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (2) and (3), is a blowup sequence, i.e*

$$u_\epsilon \rightharpoonup 0 \quad \text{weakly in } H_{1,0}^2(\Omega) \quad \text{as } \epsilon \rightarrow 0$$

We let $(x_\epsilon)_\epsilon, (\mu_\epsilon)_\epsilon$ be as in (20). Let k_ϵ be such that

$$(21) \quad k_\epsilon := |x_\epsilon|^{s_\epsilon/2} \mu_\epsilon^{\frac{2-s_\epsilon}{2}} \quad \text{for } \epsilon > 0$$

Then

$$\lim_{\epsilon \rightarrow 0} \mu_\epsilon = \lim_{\epsilon \rightarrow 0} k_\epsilon = 0 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \frac{d(x_\epsilon, \partial\Omega)}{\mu_\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} = +\infty.$$

We rescale and define

$$v_\epsilon(x) := \frac{u_\epsilon(x_\epsilon + k_\epsilon x)}{u_\epsilon(x_\epsilon)} \quad \text{for } x \in \frac{\Omega - x_\epsilon}{k_\epsilon}$$

Then there exists $v \in C^\infty(\mathbb{R}^n)$ such that $v \neq 0$ and for any $\eta \in C_c^\infty(\mathbb{R}^n)$

$$\eta v_\epsilon \rightharpoonup \eta v \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^n) \quad \text{as } \epsilon \rightarrow 0$$

and $\lim_{\epsilon \rightarrow 0} v_\epsilon = v$ in $C_{loc}^1(\mathbb{R}^n)$ where for $x \in \mathbb{R}^n$,

$$(22) \quad v(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-\frac{n-2}{2}} \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla v|^2 dx = \left(\frac{1}{K(n, 0)}\right)^{\frac{2^*}{2^*-2}}$$

Moreover upto a subsequence, as $\epsilon \rightarrow 0$

$$(23) \quad \left(\frac{\mu_\epsilon}{|x_\epsilon|}\right)^{s_\epsilon} \rightarrow 1 \quad \text{and} \quad \frac{k_\epsilon}{\mu_\epsilon} \rightarrow 1.$$

Proof. The proof goes through following steps.

Step 1: We claim that: $\mu_\epsilon = o(1)$ as $\epsilon \rightarrow 0$.

We prove our claim. Suppose $\lim_{\epsilon \rightarrow 0} \mu_\epsilon \neq 0$. Then (u_ϵ) is uniformly bounded in L^∞ , and then $(|x|^{-s} u_\epsilon^{2^*(s_\epsilon)-1})_\epsilon$ is uniformly bounded in $L^p(\Omega)$ for some $p > n$. Then from (2), the weak convergence to 0 and standard elliptic theory, we get that $u_\epsilon \rightarrow 0$ in $C^1(\overline{\Omega})$, as $\epsilon \rightarrow 0$. From (2) and (3), we then get that $\lim_{\epsilon \rightarrow 0} \mu_{s_\epsilon, a}(\Omega) = 0$ and therefore, $\mu_a(\Omega) = 0$, contradicting the coercivity. This ends Step 1.

Step 2: From Lemma 1 it follows that

$$(24) \quad \lim_{\epsilon \rightarrow 0} \frac{|x_\epsilon|}{\mu_\epsilon} = +\infty, \quad \lim_{\epsilon \rightarrow 0} \frac{d(x_\epsilon, \partial\Omega)}{k_\epsilon} = +\infty.$$

and, there exist $v \in C^1(\mathbb{R}^n)$, $v > 0$, such that $\lim_{\epsilon \rightarrow 0} v_\epsilon = v$ in $C_{loc}^1(\mathbb{R}^n)$ and it satisfies $\Delta v = v^{2^*-1}$ in \mathbb{R}^n . Further we have that $\max_{x \in \mathbb{R}^n} v(x) = v(0) = 1$. By Caffarelli, Gidas and Spruck [5], we then have the first assertion of (22).

Step 3: Arguing as in the proof of (18), for any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $R > 0$

(25)

$$\int_{\mathbb{R}^n} |\nabla(\eta_R v_\epsilon)|^2 dx \leq C(\theta) \left(\int_{B_0(2R) \setminus B_0(R)} (\eta_{2R} v_\epsilon)^{2^*} dx \right)^{\frac{n-2}{n}} + (1+\theta) \left(\frac{\mu_\epsilon}{k_\epsilon} \right)^{n-2} \int_{\Omega} |\nabla u_\epsilon|^2 dx$$

Now $u_\epsilon \rightharpoonup 0$ weakly in $H_{1,0}^2(\Omega)$ as $\epsilon \rightarrow 0$, where for each $\epsilon > 0$, u_ϵ satisfies (2) and (3). So we have

$$\int_{\Omega} |\nabla u_\epsilon|^2 dx = \int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx + o(1) \leq \mu_{s_\epsilon, a}(\Omega)^{\frac{2^*(s_\epsilon)}{2^*(s_\epsilon)-2}} + o(1) \text{ as } \epsilon \rightarrow 0.$$

Using Proposition 2.1, letting $\epsilon \rightarrow 0$, then $R \rightarrow +\infty$, and the $\theta \rightarrow 0$, we obtain

$$(26) \quad \int_{\mathbb{R}^n} |\nabla v|^2 dx \leq \left(\limsup_{\epsilon \rightarrow 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|} \right)^{s_\epsilon} \right)^{\frac{n-2}{2}} \mu_a(\Omega)^{\frac{2^*}{2^*-2}}$$

From (24) we get $\limsup_{\epsilon \rightarrow 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|} \right)^{s_\epsilon} \leq 1$. Since $\mu_a(\Omega) \leq \frac{1}{K(n,0)}$ (see Aubin [2]), we get

$$\int_{\mathbb{R}^n} |\nabla v|^2 dx \leq \mu_a(\Omega)^{\frac{2^*}{2^*-2}} \leq \left(\frac{1}{K(n,0)} \right)^{\frac{2^*}{2^*-2}}$$

Since $v \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfies $\Delta v = v^{2^*-1}$, Sobolev's inequality (5) then yields the second assertion of (22). Then (26) implies $\limsup_{\epsilon \rightarrow 0} \left(\frac{\mu_\epsilon}{|x_\epsilon|} \right)^{s_\epsilon} \geq 1$, which yields (23).

This completes the proof of Theorem 4. \square

As a consequence of Theorem 4, we get the following concentration of energy:

Proposition 3.1. *Under the hypothesis of Theorem 4 one further has that*

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx = 0$$

Proof. We obtain by change of variables

$$\begin{aligned} \int_{\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx &= \int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx - \int_{B_{x_\epsilon}(Rk_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx \\ &= \int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx - \frac{k_\epsilon^n}{\mu_\epsilon^{n-s_\epsilon}} \int_{B_0(R)} \frac{|v_\epsilon(x)|^{2^*(s_\epsilon)}}{|x_\epsilon + k_\epsilon x|^{s_\epsilon}} dx \\ &= \int_{\Omega} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx - \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} \right)^{\frac{n-2}{2}} \int_{B_0(R)} \frac{|v_\epsilon(x)|^{2^*(s_\epsilon)}}{\left| \frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|} x \right|^{s_\epsilon}} dx \end{aligned}$$

Letting $\epsilon \rightarrow 0$ and then $R \rightarrow +\infty$ one obtains the proposition using Theorem 4. \square

4. REFINED BLOWUP ANALYSIS I

In this section we obtain pointwise bounds on the blowup sequence $(u_\epsilon)_{\epsilon>0}$ that will be used in next section to get the optimal bound.

Theorem 5. *With the same hypothesis as in Theorem 4, we have that there exists a constant $C > 0$ such that for $\epsilon > 0$*

$$|x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) + \frac{|x - x_\epsilon|^{\frac{n}{2}}}{d(x, \partial\Omega)} u_\epsilon(x) \leq C \quad \text{for all } x \in \Omega.$$

Moreover,

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \sup_{x \in \Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} |x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) = 0$$

The proof of Theorem 5 comprises the three propositions proved below.

Proposition 4.1. *With the same hypothesis as in Theorem 4, we have that there exists a constant $C > 0$ such that for $\epsilon > 0$*

$$|x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) \leq C \quad \text{for all } x \in \Omega$$

Proof. We argue by contradiction and let $y_\epsilon \in \Omega$ be such that

$$(27) \quad |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(y_\epsilon) = \sup_{x \in \Omega} \left(|x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) \right) \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

Then

$$(28) \quad |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(y_\epsilon) \rightarrow +\infty \quad \text{as } \epsilon \rightarrow 0$$

We define $\lambda_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(y_\epsilon)$. Then $\mu_\epsilon \leq \lambda_\epsilon$, and (28) becomes

$$(29) \quad \lim_{\epsilon \rightarrow 0} \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} = +\infty.$$

and so we have that $\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = 0$.

Step 1: It follows from the definition (27) and (29) that given any $R > 0$ one has for $\epsilon > 0$ sufficiently small $u_\epsilon(x) \leq 2u_\epsilon(y_\epsilon)$ for all $x \in B_{y_\epsilon}(R\lambda_\epsilon)$. Therefore hypothesis (10) of Lemma 1 is satisfied and one has

$$(30) \quad \lim_{\epsilon \rightarrow 0} \frac{|y_\epsilon|}{\lambda_\epsilon} = +\infty$$

Define $l_\epsilon = |y_\epsilon|^{s_\epsilon/2} \lambda_\epsilon^{\frac{2-s_\epsilon}{2}}$ for all $\epsilon > 0$. Then $\lim_{\epsilon \rightarrow 0} l_\epsilon = 0$. Moreover, we have that

$$(31) \quad \lim_{\epsilon \rightarrow 0} \frac{l_\epsilon}{|y_\epsilon|} = \lim_{\epsilon \rightarrow 0} \left(\frac{\lambda_\epsilon}{|y_\epsilon|} \right)^{\frac{2-s_\epsilon}{2}} = 0.$$

Step 2: We claim that

$$(32) \quad \lim_{\epsilon \rightarrow 0} \frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} = +\infty.$$

We prove the claim. Due to (31), the claim is clear when $y_\epsilon = O(|y_\epsilon - x_\epsilon|)$ as $\epsilon \rightarrow 0$. We assume that $y_\epsilon - x_\epsilon = o(|y_\epsilon|)$ as $\epsilon \rightarrow 0$. We then have that $|x_\epsilon| \asymp |y_\epsilon|$ as $\epsilon \rightarrow 0$. Therefore, there exists $c_0 > 0$ such that

$$(33) \quad c_0 \leq \frac{|x_\epsilon|^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} = \frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} \frac{|x_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}} \leq \frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} \frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}}$$

Since $\lim_{\epsilon \rightarrow 0} \frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} = 1$ as shown in (23), it follows that there exists $c_1 > 0$ such that $\left(\frac{\lambda_\epsilon}{|y_\epsilon|}\right)^{s_\epsilon} \geq c_1$ for $\epsilon > 0$ small enough. Therefore,

$$(34) \quad \lim_{\epsilon \rightarrow 0} \frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} \frac{\lambda_\epsilon^{s_\epsilon/2}}{|y_\epsilon|^{s_\epsilon/2}} = +\infty$$

and the claim is proved.

Step 3: It follows from (32) and the definitions (27) and (28) that for any $R > 0$ one has for $\epsilon > 0$ sufficiently small $u_\epsilon(x) \leq 2u_\epsilon(y_\epsilon)$ for all $x \in B_{y_\epsilon}(Rl_\epsilon)$. Therefore hypothesis (11) of Lemma 1 is satisfied and one has

$$(35) \quad \lim_{\epsilon \rightarrow 0} \frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} = +\infty.$$

We let for $\epsilon > 0$

$$w_\epsilon(x) = \frac{u_\epsilon(y_\epsilon + l_\epsilon x)}{u_\epsilon(y_\epsilon)} \quad \text{for } x \in \frac{\Omega - y_\epsilon}{l_\epsilon}.$$

From Lemma 1 it follows that $\lim_{\epsilon \rightarrow 0} w_\epsilon = w$ in $C_{loc}^1(\mathbb{R}^n)$ where $w \in C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,2}(\mathbb{R}^n)$ is such that $\Delta w = w^{2^*-1}$ in \mathbb{R}^n , $w \geq 0$ and $w(0) = 1$. We obtain by a change of variable for $R > 0$ and $\epsilon > 0$

$$\int_{B_0(R)} \frac{|w_\epsilon(x)|^{2^*(s_\epsilon)}}{\left|\frac{y_\epsilon}{|y_\epsilon|} + \frac{l_\epsilon}{|y_\epsilon|}x\right|^{s_\epsilon}} dx = \left(\frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}}\right)^{\frac{n-2}{2}} \int_{B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx$$

Passing to the limit as $\epsilon \rightarrow 0$, we have for $R > 0$

$$\int_{B_0(R)} w^{2^*} dx \leq \limsup_{\epsilon \rightarrow 0} \int_{B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx$$

and so

$$\int_{\mathbb{R}^n} w^{2^*} dx = \lim_{R \rightarrow +\infty} \int_{B_0(R)} w^{2^*} dx \leq \lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx$$

Now for any $R > 0$, we claim that $B_{x_\epsilon}(Rk_\epsilon) \cap B_{y_\epsilon}(Rl_\epsilon) = \emptyset$ for $\epsilon > 0$ sufficiently small. We argue by contardiction and we assume that the intersection is nonempty, which yields $y_\epsilon - x_\epsilon = O(k_\epsilon + l_\epsilon)$ as $\epsilon \rightarrow 0$, up to extraction. It then follows from (32) that $y_\epsilon - x_\epsilon = O(k_\epsilon)$ as $\epsilon \rightarrow 0$, and then $|y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(y_\epsilon) = O(k_\epsilon^{\frac{n-2}{2}} u_\epsilon(y_\epsilon)) = O(\mu_\epsilon^{\frac{n-2}{2}} u_\epsilon(x_\epsilon)) = O(1)$ with (20) and (23). This contradicts (28) and proves the claim. Then by Proposition 3.1

$$\int_{\mathbb{R}^n} w^{2^*} dx \leq \lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_{y_\epsilon}(Rl_\epsilon)} \frac{|u_\epsilon(x)|^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} dx = 0$$

A contradiction since $w(0) = 1$. Hence (28) does not hold. This completes the proof of Proposition 4.1. \square

Having obtained the strong bound in Proposition 4.1 we show that

Proposition 4.2. *With the same hypothesis as in Theorem 4 we have that there exists a constant $C > 0$ such that for $\epsilon > 0$*

$$|x - x_\epsilon|^{n/2} |\nabla u_\epsilon(x)| \leq C \quad \text{and} \quad |x - x_\epsilon|^{n/2} u_\epsilon(x) \leq Cd(x, \partial\Omega) \quad \text{for all } x \in \Omega$$

Proof. We proceed by contradiction and assume that there exists a sequence of points $(y_\epsilon)_{\epsilon > 0}$ in Ω such that

$$(36) \quad |y_\epsilon - x_\epsilon|^{n/2} |\nabla u_\epsilon(y_\epsilon)| + \frac{|y_\epsilon - x_\epsilon|^{n/2} u_\epsilon(y_\epsilon)}{d(y_\epsilon, \partial\Omega)} \longrightarrow +\infty \quad \text{as } \epsilon \rightarrow 0$$

We define $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in \overline{\Omega}$ and $\lim_{\epsilon \rightarrow 0} y_\epsilon = y_0 \in \overline{\Omega}$.

Case 1: we assume that $x_0 \neq y_0$. We choose $\delta > 0$ such that $0 < 4\delta < |x_0 - y_0|$. Then one has that $\delta < |x - x_\epsilon|$ for all $x \in B_{y_0}(2\delta) \cap \Omega$ and Lemma 4.1 then gives us that there exists a constant $C(\delta) > 0$ such that $0 \leq u_\epsilon \leq C(\delta)$ in $B_{y_0}(2\delta)$. Then from equation (2) and standard elliptic theory, u_ϵ is bounded in $C^1(B_{y_0}(\delta) \cap \overline{\Omega})$. So there exists a constant $C > 0$ such that $|\nabla u_\epsilon(x)| \leq C$ and $u_\epsilon(x) \leq Cd(x, \partial\Omega)$ for all $x \in B_{y_0}(\delta) \cap \overline{\Omega}$. This contradicts (36). The proposition is proved in Case 1.

Case 2: we assume that $x_0 = y_0$. Define $\alpha_\epsilon = |y_\epsilon - x_\epsilon|$, so that $\lim_{\epsilon \rightarrow 0} \alpha_\epsilon = 0$.

Case 2.1: We assume that upto a subsequence

$$d(x_\epsilon, \partial\Omega) \geq 2|y_\epsilon - x_\epsilon|$$

For $\epsilon > 0$ we let

$$\tilde{u}_\epsilon(x) = \alpha_\epsilon^{\frac{n-2}{2}} u_\epsilon(x_\epsilon + \alpha_\epsilon x) \quad \text{for } x \in B_0(3/2)$$

This is well defined since $B_{x_\epsilon}(2\alpha_\epsilon) \subset \Omega$. Using Lemma 4.1 one obtains that there exists a constant $C > 0$ such that

$$|x|^{\frac{n-2}{2}} \tilde{u}_\epsilon(x) \leq C \quad \text{for } x \in B_0(3/2).$$

Arguing as in Step 1.3 of the proof of Lemma 4.1, standard elliptic theory yields

$$\|\tilde{u}_\epsilon\|_{C^1(B_0(5/4) \setminus \overline{B_0(1/2)})} = O(1) \quad \text{as } \epsilon \rightarrow 0$$

Then one then obtains as $\epsilon \rightarrow 0$

$$\left| \nabla \tilde{u}_\epsilon \left(\frac{y_\epsilon - x_\epsilon}{|y_\epsilon - x_\epsilon|} \right) \right| = O(1) \quad \text{and} \quad \tilde{u}_\epsilon \left(\frac{y_\epsilon - x_\epsilon}{|y_\epsilon - x_\epsilon|} \right) = O(1).$$

coming back to the definition of \tilde{u}_ϵ , this contradicts (36). This ends Case 2.1.

Case 2.2: We assume that upto a subsequence

$$d(x_\epsilon, \partial\Omega) \leq 2|y_\epsilon - x_\epsilon|$$

Let $\mathcal{T} : U \rightarrow V$ be a parametrisation of the boundary $\partial\Omega$ as in (8) around the point $p = x_0$. Let $z_\epsilon \in \partial\Omega$ be such that $|z_\epsilon - x_\epsilon| = d(x_\epsilon, \partial\Omega)$ for $\epsilon > 0$. We let $\tilde{x}_\epsilon, \tilde{z}_\epsilon \in U$ be such that $\mathcal{T}(\tilde{x}_\epsilon) = x_\epsilon$ and $\mathcal{T}(\tilde{z}_\epsilon) = z_\epsilon$. Then it follows from the properties of the

boundary chart \mathcal{T} , that $\lim_{\epsilon \rightarrow 0} \tilde{x}_\epsilon = 0 = \lim_{\epsilon \rightarrow 0} \tilde{z}_\epsilon$, $(\tilde{x}_\epsilon)_1 < 0$ and $(\tilde{z}_\epsilon)_1 = 0$. For all $\epsilon > 0$, we let

$$\tilde{u}_\epsilon(x) = \alpha_\epsilon^{\frac{n-2}{2}} u_\epsilon \circ \mathcal{T}(\tilde{z}_\epsilon + \alpha_\epsilon x) \quad \text{for } x \in \frac{U - \tilde{z}_\epsilon}{\alpha_\epsilon} \cap \{x_1 \leq 0\}$$

For any $R > 0$, \tilde{u}_ϵ is defined in $B_0(R) \cap \{x_1 \leq 0\}$ for $\epsilon > 0$ small enough. Using lemma Lemma 4.1 and the properties of the chart \mathcal{T} , one obtains that there exists a constant $C > 0$ such that

$$|\rho_\epsilon - x|^{\frac{n-2}{2}} \tilde{u}_\epsilon(x) \leq C \quad \text{for } x \in B_0(R) \cap \{x_1 \leq 0\}$$

where $\rho_\epsilon = \frac{\tilde{x}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon}$ and there exists $\rho_0 \in \overline{\mathbb{R}}_-$ such that $\rho_\epsilon \rightarrow \rho_0$ as $\epsilon \rightarrow 0$. Arguing again as in Step 1.3 of the proof of Lemma 1, standard elliptic theory yields

$$\|\tilde{u}_\epsilon\|_{C^1(\overline{B_0(R/2)} \setminus B_{\rho_0}(2\delta) \cap \{x_1 \leq 0\})} = O(1) \quad \text{as } \epsilon \rightarrow 0$$

and \tilde{u}_ϵ vanishes on the boundary $B_0(R/2) \setminus \overline{B_{\rho_0}(2\delta)} \cap \{x_1 = 0\}$. Let $\tilde{y}_\epsilon \in U$ be such that $\mathcal{T}(\tilde{y}_\epsilon) = y_\epsilon$. It then follows that, as $\epsilon \rightarrow 0$

$$\left| \nabla \tilde{u}_\epsilon \left(\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} \right) \right| = O(1), \quad \tilde{u}_\epsilon \left(\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} \right) = O(1)$$

and since \tilde{u}_ϵ vanishes on the boundary $B_0(R/2) \setminus \overline{B_{\rho_0}(2\delta)} \cap \{x_1 = 0\}$, it follows that

$$0 \leq \tilde{u}_\epsilon \left(\frac{\tilde{y}_\epsilon - \tilde{z}_\epsilon}{\alpha_\epsilon} \right) = O \left(\frac{(\tilde{y}_\epsilon - \tilde{z}_\epsilon)_1}{\alpha_\epsilon} \right) = O \left(\frac{(\tilde{y}_\epsilon)_1}{\alpha_\epsilon} \right) = O \left(\frac{d(y_\epsilon, \partial\Omega)}{\alpha_\epsilon} \right)$$

comig back to the definition of \tilde{u}_ϵ this implies that as $\epsilon \rightarrow 0$

$$|y_\epsilon - x_\epsilon|^{n/2} |\nabla u_\epsilon(y_\epsilon)| = O(1), \quad \text{and} \quad |y_\epsilon - x_\epsilon|^{n/2} u_\epsilon(y_\epsilon) = O(d(x_\epsilon, \partial\Omega)),$$

contradicting (36). This ends Case 2.2.

All these cases prove Proposition 4.2. \square

As a consequence of Proposition 4.1 and Proposition 4.2 we get the following:

Corollary 4.1. *Let $(u_\epsilon)_{\epsilon>0}$ be as in Theorem 4, and let $\lim_{\epsilon \rightarrow 0} x_\epsilon \rightarrow x_0 \in \overline{\Omega}$, then upto a subsequence $\lim_{\epsilon \rightarrow 0} u_\epsilon = 0$ in $C_{loc}^1(\overline{\Omega} \setminus \{x_0\})$.*

We slightly improve our estimate in Proposition 4.1 to obtain

Proposition 4.3. *With the same hypothesis as in Theorem 4 we have*

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \sup_{x \in \Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} |x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) = 0$$

Proof. Suppose on the contrary there exists $\epsilon_0 > 0$ and a sequence of points $(y_\epsilon)_{\epsilon>0} \in \Omega$ such that upto a subsequence

$$(37) \quad |y_\epsilon - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(y_\epsilon) \geq \epsilon_0^{\frac{n-2}{2}} \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \frac{|y_\epsilon - x_\epsilon|}{k_\epsilon} = +\infty$$

It then follows from Corollary 4.1 that $\lim_{\epsilon \rightarrow 0} |y_\epsilon - x_\epsilon| = 0$. We define $\lambda_\epsilon^{-\frac{n-2}{2}} = u_\epsilon(y_\epsilon)$. Then (37) becomes

$$(38) \quad C \geq \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} \geq \epsilon_0 \quad \text{for all } \epsilon > 0$$

and so $\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = 0$. using Lemma 4.1 we obtain that as $\epsilon \rightarrow 0$

$$(39) \quad \frac{k_\epsilon}{\lambda_\epsilon} = \frac{k_\epsilon}{|y_\epsilon - x_\epsilon|} \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} = O\left(\frac{k_\epsilon}{|y_\epsilon - x_\epsilon|}\right) = o(1)$$

We define $l_\epsilon = |y_\epsilon|^{s_\epsilon/2} \lambda_\epsilon^{\frac{2-s_\epsilon}{2}}$ for $\epsilon > 0$. Then $\lim_{\epsilon \rightarrow 0} l_\epsilon = 0$.

We first claim that

$$(40) \quad \frac{|y_\epsilon|^{s_\epsilon}}{\lambda_\epsilon^{s_\epsilon}} = O(1) \quad \text{as } \epsilon \rightarrow 0$$

We proceed by contradiction and we assume that $\lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} = 0$. Now, using (33) and (23), we get that $\lim_{\epsilon \rightarrow 0} \frac{|x_\epsilon|^{s_\epsilon}}{|y_\epsilon|^{s_\epsilon}} = 0$. And in particular one has that $\lim_{\epsilon \rightarrow 0} \frac{|x_\epsilon|}{|y_\epsilon|} = 0$ and $\lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{|y_\epsilon|} = 0$. Then

$$\frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} \geq \frac{|y_\epsilon|}{\lambda_\epsilon} \left| 1 - \frac{|x_\epsilon|}{|y_\epsilon|} \right| \rightarrow +\infty \text{ as } \epsilon \rightarrow 0$$

A contradiction to (38), proving our claim. We note that then there exists $c_2 > 0$ such that for $\epsilon > 0$ small

$$(41) \quad \frac{|y_\epsilon - x_\epsilon|}{l_\epsilon} = \frac{|y_\epsilon - x_\epsilon|}{\lambda_\epsilon} \frac{\lambda_\epsilon^{s_\epsilon/2}}{|y_\epsilon|^{s_\epsilon/2}} \geq c_2$$

Arguing as in case 2.2 of Lemma 4.1 we see that we cannot have $\lim_{\epsilon \rightarrow 0} \frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} = 0$.

Let $\rho_0 > 0$ be such that upto a subsequence $\frac{d(y_\epsilon, \partial\Omega)}{l_\epsilon} \geq 2\rho_0$. Without loss of generality we can take $2\rho_0 < c_2$. Then proceeding as in step 3 of Lemma 4.1 we arrive at a contradiction. These steps complete the proof of Proposition 4.3. \square

5. REFINED BLOWUP ANALYSIS II

This section is devoted to the proof of Theorem 1.

Proof. Step 1: We claim that for any $\alpha \in (0, n-2)$, there exists $C_\alpha > 0$ such that for all $\epsilon > 0$

$$(42) \quad |x - x_\epsilon|^\alpha \mu_\epsilon^{\frac{n-2}{2}-\alpha} u_\epsilon(x) \leq C_\alpha \quad \text{for all } x \in \Omega$$

Proof. Since the operator $\Delta + a$ is coercive on Ω and $a \in C(\overline{\Omega})$, there exists $U_0 \subset \mathbb{R}^n$ an open set such that $\overline{\Omega} \subset \subset U_0$, and there exists $a_1 > 0$, $A_1 > 0$ such that

$$\int_{U_0} |\nabla \varphi|^2 dx + \int_{U_0} (a - a_1) \varphi^2 dx \geq A_1 \int_{U_0} \varphi^2 dx \quad \text{for all } \varphi \in C_c^\infty(U_0),$$

where we have continuously extended a to U_0 . In other words the operator $\Delta + (a - a_1)$ is coercive on U_0 . Let $\tilde{G} : \overline{U_0} \times \overline{U_0} \setminus \{(x, x) : x \in \overline{U_0}\} \rightarrow \mathbb{R}$ be the *Green's function* of the operator $\Delta + (a - a_1)$ with Dirichlet boundary conditions. The \tilde{G} satisfies

$$(43) \quad \Delta \tilde{G}(x, \cdot) + (a - a_1) \tilde{G}(x, \cdot) = \delta_x$$

Since the operator $\Delta + (a - a_1)$ is coercive on U_0 , \tilde{G} exists. See Robert [19]. We set $\tilde{G}_\epsilon(x) = \tilde{G}(x_\epsilon, x)$ for all $x \in \overline{U_0} \setminus \{x_\epsilon\}$ and $\epsilon > 0$. Then there exists $C > 0$ such that

$$0 < \tilde{G}_\epsilon(x) < \frac{C}{|x - x_\epsilon|^{n-2}} \quad \text{for } x \in \overline{U_0} \setminus \{x_\epsilon\}.$$

Moreover there exists $\delta_0 > 0$ and $C_0 > 0$ such that for all $\epsilon > 0$

(44)

$$\tilde{G}_\epsilon(x) \geq \frac{C_0}{|x - x_\epsilon|^{n-2}} \quad \text{and} \quad \frac{|\nabla \tilde{G}_\epsilon(x)|}{|\tilde{G}_\epsilon(x)|} \geq \frac{C_0}{|x - x_\epsilon|} \quad \text{for } x \in B_{x_\epsilon}(\delta_0) \setminus \{x_\epsilon\} \subset \subset U_0$$

We define the operator

$$\mathcal{L}_\epsilon = \Delta + a - \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}}$$

Step 1.1: We claim that there exists $\nu_0 \in (0, 1)$ such that given any $\nu \in (0, \nu_0)$ there exists $R_1 > 0$ such that for $R > R_1$ and $\epsilon > 0$ sufficiently small we have

$$(45) \quad \mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu} > 0 \quad \text{in } \Omega \setminus B_{x_\epsilon}(Rk_\epsilon)$$

We prove the claim. We choose $\nu_0 \in (0, 1)$ such that for any $\nu \in (0, \nu_0)$ one has $\nu(a - a_1) \geq -\frac{a_1}{2}$ in Ω . Fix $\nu \in (0, \nu_0)$. Using (43) we obtain for $\epsilon > 0$ sufficiently small

$$\begin{aligned} \frac{\mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu}}{\tilde{G}_\epsilon^{1-\nu}} &= a_1 + \nu(a - a_1) + \nu(1 - \nu) \frac{|\nabla \tilde{G}_\epsilon|^2}{|\tilde{G}_\epsilon|^2} - \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} \quad \text{in } \Omega \setminus \{x_\epsilon\} \\ &\geq \frac{a_1}{2} + \nu(1 - \nu) \frac{|\nabla \tilde{G}_\epsilon|^2}{|\tilde{G}_\epsilon|^2} - \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} \quad \text{in } \Omega \setminus \{x_\epsilon\} \end{aligned}$$

Let $|x - x_\epsilon| \geq \delta_0$, where δ_0 is as in (44), then from Corollary 4.1 we have

$$\lim_{\epsilon \rightarrow 0} \frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} = 0 \quad \text{in } C(\overline{\Omega \setminus B_{x_\epsilon}(\delta_0)})$$

Hence for $\epsilon > 0$ sufficiently small we have for $\nu \in (0, \nu_0)$

$$\frac{\mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu}}{\tilde{G}_\epsilon^{1-\nu}} > 0 \quad \text{for } x \in \Omega \setminus B_{x_\epsilon}(\delta_0)$$

By strong pointwise estimates, Proposition 4.3 we have that, given any $\nu \in (0, \nu_0)$, there exists $R_1 > 0$ such that for any $R > R_1$

$$\sup_{\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)} |x - x_\epsilon|^{\frac{n-2}{2}} u_\epsilon(x) \leq \left[\frac{\nu(1-\nu)}{4} C_0^2 \right]^{\frac{n-2}{4}}$$

Here C_0 is as in (44). And then using Lemma 4.2 we obtain for $\epsilon > 0$ small

$$\frac{u_\epsilon^{2^*(s_\epsilon)-2}}{|x|^{s_\epsilon}} = \left[u_\epsilon^{2^*(s_\epsilon)-2-s_\epsilon} \left(\frac{u_\epsilon}{|x|} \right)^{s_\epsilon} \right] \leq \frac{\nu(1-\nu)}{2} \frac{C_0^2}{|x - x_\epsilon|^2}$$

for all $x \in \Omega \setminus B_{x_\epsilon}(Rk_\epsilon)$. Therefore if $x \in B_{x_\epsilon}(\delta_0) \setminus B_{x_\epsilon}(Rk_\epsilon)$ then with (44) we get

$$\frac{\mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu}}{\tilde{G}_\epsilon^{1-\nu}} \geq \frac{a_1}{2} + \frac{\nu(1-\nu)}{2} \frac{C_0^2}{|x - x_\epsilon|^2} > 0$$

for $\epsilon > 0$ small. This proves the claim and ends Step 1.1.

Step 1.2: Let $\nu \in (0, \nu_0)$ and $R > R_1$. We claim that there exists $C(R) > 0$ such that for $\epsilon > 0$ small

$$(46) \quad \begin{aligned} \mathcal{L}_\epsilon \left(C(R) \mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \tilde{G}_\epsilon^{1-\nu} \right) &> \mathcal{L}_\epsilon u_\epsilon \quad \text{in } \Omega \setminus B_{x_\epsilon}(Rk_\epsilon) \\ C(R) \mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \tilde{G}_\epsilon^{1-\nu} &> u_\epsilon \quad \text{on } \partial(\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)) \end{aligned}$$

We prove the claim. Since $\mathcal{L}_\epsilon u_\epsilon = 0$ in Ω , so it follows from (45) that $\mathcal{L}_\epsilon \left(C(R) \mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \tilde{G}_\epsilon^{1-\nu} \right) > \mathcal{L}_\epsilon u_\epsilon$ in $\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)$ for $R > R_1$ and $\epsilon > 0$ sufficiently small. With (44) and (23), we obtain for $\epsilon > 0$ small

$$\frac{u_\epsilon(x)}{\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \tilde{G}_\epsilon^{1-\nu}(x)} \leq \frac{\mu_\epsilon^{-\frac{n-2}{2}} (Rk_\epsilon)^{(n-2)(1-\nu)}}{\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} C_0^{1-\nu}} \leq \frac{(2R)^{(n-2)(1-\nu)}}{C_0^{1-\nu}}$$

for $x \in \Omega \cap \partial B_{x_\epsilon}(Rk_\epsilon)$. So for $x \in \partial(\Omega \setminus B_{x_\epsilon}(Rk_\epsilon))$ one has for $\epsilon > 0$ small

$$\frac{u_\epsilon(x)}{\mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \tilde{G}_\epsilon^{1-\nu}(x)} \leq C(R) \quad \text{for } x \in \Omega \cap \partial B_{x_\epsilon}(Rk_\epsilon)$$

This proves the claim and ends Step 1.2.

Step 1.3: Let $\nu \in (0, \nu_0)$ and $R > R_1$. Since $\tilde{G}_\epsilon^{1-\nu} > 0$ in $\overline{\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)}$ and $\mathcal{L}_\epsilon \tilde{G}_\epsilon^{1-\nu} > 0$ in $\Omega \setminus B_{x_\epsilon}(Rk_\epsilon)$, it follows from [3] that the operator \mathcal{L}_ϵ satisfies the comparison principle. Then from (46) we have that for $\epsilon > 0$ small

$$u_\epsilon(x) \leq C(R) \mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \tilde{G}_\epsilon^{1-\nu}(x) \quad \text{for } x \in \Omega \setminus B_{x_\epsilon}(Rk_\epsilon)$$

Then with (44) we get that

$$|x - x_\epsilon|^{(n-2)(1-\nu)} u_\epsilon(x) \leq C(R) \mu_\epsilon^{\frac{n-2}{2}-\nu(n-2)} \quad \text{for } x \in \Omega \setminus B_{x_\epsilon}(Rk_\epsilon)$$

Taking $\alpha = (n-2)(1-\nu)$, we have for α close to $n-2$

$$|x - x_\epsilon|^\alpha \mu_\epsilon^{\frac{n-2}{2}-\alpha} u_\epsilon(x) \leq C_\alpha \quad \text{for } x \in \Omega \setminus B_{x_\epsilon}(Rk_\epsilon).$$

As easily checked, this implies (42) for all $\alpha \in (0, n-2)$. This ends Step 1.3 and also Step 1. \square

Next we show that one can infact take $\alpha = n-2$ in (42).

Step 2: We claim that there exists $C > 0$ such that for all $\epsilon > 0$

$$(47) \quad |x - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) u_\epsilon(x) \leq C \quad \text{for all } x \in \Omega$$

Proof. The claim is equivalent to proving that for any $(y_\epsilon)_\epsilon \in \Omega$, we have that

$$|y_\epsilon - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) = O(1) \quad \text{as } \epsilon \rightarrow 0$$

We have the following two cases.

Step 2.1: Suppose that $|x_\epsilon - y_\epsilon| = O(\mu_\epsilon)$ as $\epsilon \rightarrow 0$. By definition (20) it follows that $|y_\epsilon - x_\epsilon|^{n-2} u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) \leq |y_\epsilon - x_\epsilon|^{n-2} \mu_\epsilon^{2-n}$. This proves (47) in this case and ends Step 2.1.

Step 2.2: Suppose that

$$(48) \quad \lim_{\epsilon \rightarrow 0} \frac{|x_\epsilon - y_\epsilon|}{\mu_\epsilon} = +\infty \quad \text{as } \epsilon \rightarrow 0$$

We let for $\epsilon > 0$

$$\hat{v}_\epsilon(x) = \mu_\epsilon^{\frac{n-2}{2}} u_\epsilon(\mu_\epsilon x + x_\epsilon) \quad \text{for } x \in \frac{\Omega - x_\epsilon}{\mu_\epsilon}$$

Then from (42), it follows that for any $\alpha \in (0, n-2)$, there exists $C'_\alpha > 0$ such that for all $\epsilon > 0$

$$\hat{v}_\epsilon(x) \leq \frac{C'_\alpha}{1 + |x|^\alpha} \quad \text{for } x \in \frac{\Omega - x_\epsilon}{\mu_\epsilon}$$

Let G be the Green's function of $\Delta + a$ with Dirichlet boundary conditions. Green's representation formula and standard estimates on the Green's function yield

$$u_\epsilon(y_\epsilon) = \int_{\Omega} G(x, y_\epsilon) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} dx \leq C \int_{\Omega} \frac{1}{|x - y_\epsilon|^{n-2}} \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} dx \quad \text{for all } \epsilon > 0$$

where $C > 0$ is a constant. We write the above integral as follows

$$u_\epsilon(y_\epsilon) \leq C \int_{\Omega} \left(\frac{u_\epsilon(x)}{|x|} \right)^{s_\epsilon} \frac{1}{|x - y_\epsilon|^{n-2}} u_\epsilon(x)^{2^*(s_\epsilon)-1-s_\epsilon} dx \quad \text{for all } \epsilon > 0$$

Using Hölder inequality and then by Hardy inequality (6) we get that for $\epsilon > 0$

$$\begin{aligned} u_\epsilon(y_\epsilon) &\leq C \left(\int_{\Omega} \frac{|u_\epsilon(x)|^2}{|x|^2} dx \right)^{s_\epsilon/2} \left(\int_{\Omega} \left(\frac{1}{|x - y_\epsilon|^{n-2}} \right)^{\frac{2}{2-s_\epsilon}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx \right)^{\frac{2-s_\epsilon}{2}} \\ &\leq C \left(\left(\frac{2}{n-2} \right)^2 \int_{\Omega} |\nabla u_\epsilon|^2 dx \right)^{s_\epsilon/2} \left(\int_{\Omega} \left(\frac{1}{|x - y_\epsilon|^{n-2}} \right)^{\frac{2}{2-s_\epsilon}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx \right)^{\frac{2-s_\epsilon}{2}} \end{aligned}$$

Since $(u_\epsilon)_{\epsilon>0}$ is bounded in $H_{1,0}^2(\Omega)$, there exists $C > 0$ such that for $\epsilon > 0$ small

$$u_\epsilon(y_\epsilon)^{\frac{2}{2-s_\epsilon}} \leq C \int_{\Omega} \frac{1}{|x - y_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}} u_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx$$

With a change of variables the above integral becomes

$$u_\epsilon(y_\epsilon)^{\frac{2}{2-s_\epsilon}} \leq C \frac{\mu_\epsilon^n}{\mu_\epsilon^{\frac{n-2}{2-s_\epsilon}(2^*(s_\epsilon)-1-s_\epsilon)}} \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx$$

And so we get that for $\epsilon > 0$ small

$$\begin{aligned} \left(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_\epsilon) \right)^{\frac{2}{2-s_\epsilon}} &\leq C \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{ |y_\epsilon - x_\epsilon - \mu_\epsilon x| \geq \frac{|y_\epsilon - x_\epsilon|}{2} \right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx \\ (49) \quad &+ C \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \left\{ |y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2} \right\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx \end{aligned}$$

We estimate the above two integrals separately. First we have for $\epsilon > 0$ small and α close to $n - 2$

$$(50) \quad \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \geq \frac{|y_\epsilon - x_\epsilon|}{2}\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx \\ \leq \frac{2^{\frac{2(n-2)}{2-s_\epsilon}}}{|y_\epsilon - x_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}} \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx = O\left(\frac{1}{|y_\epsilon - x_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}}\right)$$

as $\epsilon \rightarrow 0$. On the other hand for $\epsilon > 0$ small

$$\int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \hat{v}_\epsilon(x)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2}{2-s_\epsilon}} dx \\ \leq C_\alpha \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} \frac{1}{|x|^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}}} dx \\ \leq C_\alpha \left(\frac{2\mu_\epsilon}{|y_\epsilon - x_\epsilon|}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}} \int_{\{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\}} \frac{1}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{\frac{2(n-2)}{2-s_\epsilon}}} dx \\ \leq C_\alpha \left(\frac{\mu_\epsilon}{|y_\epsilon - x_\epsilon|}\right)^{(2^*(s_\epsilon)-1-s_\epsilon)\frac{2\alpha}{2-s_\epsilon}-n} \left(\frac{1}{|y_\epsilon - x_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}}$$

Taking α close to $(n - 2)$, and using (48), we obtain for ϵ sufficiently small

$$(51) \quad \int_{\frac{\Omega - x_\epsilon}{\mu_\epsilon} \cap \{|y_\epsilon - x_\epsilon - \mu_\epsilon x| \leq \frac{|y_\epsilon - x_\epsilon|}{2}\}} \frac{\hat{v}_\epsilon^{2^*(s_\epsilon)-1}(x)}{|y_\epsilon - x_\epsilon - \mu_\epsilon x|^{n-2}} dx = o\left(\frac{1}{|y_\epsilon - x_\epsilon|^{n-2}}\right)^{\frac{2}{2-s_\epsilon}}$$

as $\epsilon \rightarrow 0$. Combining (49), (50) and (51) we obtain that

$$\left(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_\epsilon)\right)^{\frac{2}{2-s_\epsilon}} \leq O\left(\frac{1}{|y_\epsilon - x_\epsilon|^{\frac{2(n-2)}{2-s_\epsilon}}}\right) \quad \text{as } \epsilon \rightarrow 0$$

This proves (47) and ends Step 2.2 and then Step 2. \square

Step 3: The estimate (47) and the definition (20) of μ_ϵ yield Theorem 1. \square

6. LOCALIZING THE SINGULARITY: THE INTERIOR BLOW-UP CASE

In this section we prove Theorem 2. We assume that

$$x_0 \in \Omega.$$

The proof goes through four steps. We first recall the Pohozaev identity. Let U be a bounded smooth domain in \mathbb{R}^n , let $p_0 \in \mathbb{R}^n$ be a point and let $u \in C^2(\bar{U})$. We have

$$(52) \quad \int_U \left((x - p_0, \nabla u) + \frac{n-2}{2} u \right) \Delta u \, dx = \int_{\partial U} \left((x - p_0, \nu) \frac{|\nabla u|^2}{2} - \left((x - p_0, \nabla u) + \frac{n-2}{2} u \right) \partial_\nu u \right) d\sigma$$

here ν is the outer normal to the boundary ∂U . Using the above Pohozaev Identity we obtain the following identity for the Hardy Sobolev equation: Let U_ϵ be a family of smooth domains such that $x_\epsilon \in U_\epsilon \subset \Omega$ for all $\epsilon > 0$. One has for all $\epsilon > 0$

$$(53) \quad \int_{U_\epsilon} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2} \right) u_\epsilon^2 dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{U_\epsilon} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon}} \frac{1}{|x|^2} dx =$$

$$\int_{\partial U_\epsilon} (x - x_\epsilon, \nu) \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) d\sigma - \int_{\partial U_\epsilon} \left((x - x_\epsilon, \nabla u_\epsilon) + \frac{n-2}{2} u_\epsilon \right) \partial_\nu u_\epsilon d\sigma$$

Since $x_0 \in \Omega$, let $\delta > 0$ be such that $B_{x_0}(3\delta) \subset \Omega$. Note that then $\lim_{\epsilon \rightarrow 0} |x_\epsilon|^{s_\epsilon} = 1$, and it follows from (23) that $\lim_{\epsilon \rightarrow 0} \mu_\epsilon^{s_\epsilon} = 1$. We will estimate each of the terms in the above Pohozaev identity and calculate the limit as $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$. It will depend on the dimension n .

Step 1: We prove the following convergence outside x_0 :

Proposition 6.1. *We have that $\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon \rightarrow b_n G_{x_0}$ in $C_{loc}^1(\overline{\Omega} \setminus \{x_0\})$ as $\epsilon \rightarrow 0$, where b_n is as in (4) and G is the Green's function for $\Delta + a$ with Dirichlet condition.*

Proof. We fix $y_0 \in \Omega$ such that $y_0 \neq x_0$. We first claim that

$$\lim_{\epsilon \rightarrow 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) \rightarrow b_n G_{x_0}(y_0).$$

We prove the claim. We choose $\delta' \in (0, \delta)$ such that $|x_0 - y_0| \geq 3\delta'$ and $|x_0| \geq 3\delta'$. From Green's representation formula we have

$$\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = \mu_\epsilon^{-\frac{n-2}{2}} \int_{B_{x_\epsilon}(\delta')} G(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} dx + \mu_\epsilon^{-\frac{n-2}{2}} \int_{\Omega \setminus B_{x_\epsilon}(\delta')} G(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} dx$$

Using the bounds on u_ϵ obtained in Theorem 1 and the estimates on the Green's function G we get as $\epsilon \rightarrow 0$

$$\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = \mu_\epsilon^{-\frac{n-2}{2}} \int_{B_{x_\epsilon}(\delta')} G(x, y_0) \frac{u_\epsilon^{2^*(s_\epsilon)-1}(x)}{|x|^{s_\epsilon}} dx + O(\mu_\epsilon^{2-s_\epsilon}).$$

Recall our definition of v_ϵ in Theorem 4. With a change of variable, Theorem 4 yields

$$\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} \right)^{\frac{n-2}{2}} \int_{B_0(\delta' k_\epsilon^{-1})} G(x_\epsilon + k_\epsilon x, y_0) \frac{v_\epsilon^{2^*(s_\epsilon)-1}(x)}{\left| \frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|} x \right|^{s_\epsilon}} dx + O(\mu_\epsilon^{2-s_\epsilon})$$

Lebesgue dominated convergence theorem, Theorems 4 and 1 then yield

$$(54) \quad \lim_{\epsilon \rightarrow 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_0) = G(x_0, y_0) \int_{\mathbb{R}^n} v^{2^*-1} dx = b_n G(x_0, y_0).$$

This proves the claim. From (2), we get that

$$\begin{aligned} \Delta(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon) + a(x)(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon) &= \mu_\epsilon^{2-s_\epsilon} \frac{(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)^{2^*(s_\epsilon)-1}}{|x|^{s_\epsilon}} \quad \text{in } \Omega \\ \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

It follows from the pointwise estimate of Theorem 1 that $\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon$ is uniformly bounded in $L_{loc}^\infty(\Omega \setminus \{x_0\})$. It then follows from standard elliptic theory that the limit (54) holds in $C_{loc}^1(\overline{\Omega} \setminus \{x_0\})$. This completes the proof of Proposition 6.1. \square

Step 2: Next we show that

$$(55) \quad \lim_{\epsilon \rightarrow 0} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon}} \frac{1}{|x|^2} dx = \left(\frac{1}{K(n, 0)} \right)^{\frac{2^*}{2^*-2}}.$$

Proof. Recall our definition of v_ϵ in Theorem 4. With a change of variable we have

$$\int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon}} \frac{1}{|x|^2} dx = \left(\frac{|x_\epsilon|^{s_\epsilon}}{\mu_\epsilon^{s_\epsilon}} \right)^{\frac{n-2}{2}} \int_{B_0(\delta/k_\epsilon)} \frac{(x_\epsilon + k_\epsilon x, x_\epsilon)}{|x_\epsilon + k_\epsilon x|^2} \frac{v_\epsilon(x)^{2^*(s_\epsilon)}}{\left| \frac{x_\epsilon}{|x_\epsilon|} + \frac{k_\epsilon}{|x_\epsilon|} x \right|^{s_\epsilon}} dx$$

Passing to limits, and using Theorems 4 and 1 we obtain by Lebesgue dominated convergence theorem

$$\lim_{\epsilon \rightarrow 0} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon}} \frac{1}{|x|^2} dx = \int_{\mathbb{R}^n} v^{2^*} dx = \left(\frac{1}{K(n, 0)} \right)^{\frac{2^*}{2^*-2}}.$$

This proves (55) and ends Step 2. \square

Step 3: We define $a_\epsilon(x) := a(x) + \frac{1}{2}(x - x_\epsilon, \nabla a)$ for $x \in \Omega$. We claim that

$$(56) \quad \int_{B_{x_\epsilon}(\delta)} a_\epsilon u_\epsilon^2 dx = \begin{cases} O(\delta \mu_\epsilon) & \text{for } n = 3 \text{ or } a \equiv 0, \\ \mu_\epsilon^2 \log\left(\frac{1}{k_\epsilon}\right) [64\omega_3 a(x_0) + o(1)] & \text{for } n = 4, \\ \mu_\epsilon^2 [d_n a(x_0) + o(1)] & \text{for } n \geq 5. \end{cases}$$

as $\epsilon \rightarrow 0$, where d_n is as in (4).

Proof. We divide the proof in three steps.

Case 3.1: We assume that $n \geq 5$. Recall our definition of v_ϵ in Theorem 4. With a change of variable we obtain

$$\mu_\epsilon^{-2} \int_{B_{x_\epsilon}(\delta)} a_\epsilon u_\epsilon^2 dx = \left(\frac{k_\epsilon}{\mu_\epsilon} \right)^4 \int_{B_0(\delta/k_\epsilon)} a_\epsilon(x_\epsilon + k_\epsilon x) v_\epsilon^2 dx.$$

Theorem 1 reads $v_\epsilon(x) \leq C(1 + |x|^2)^{1-n/2}$. Therefore, Lebesgue's theorem and Theorem 4 yield (56) when $n \geq 5$.

Case 3.2: We assume that $n = 4$ and we argue as in Case 3.1. With the pointwise control of Theorem 1, we get that

$$\int_{B_0(\delta/k_\epsilon)} a_\epsilon(x_\epsilon + k_\epsilon x) v_\epsilon^2 dx = \log(\delta/k_\epsilon) (64\omega_3 a(x_0) + o(1)) \text{ as } \epsilon \rightarrow 0.$$

Case 3.3: we assume that $n = 3$. It follows from Theorem 1 that there exists $C > 0$ such that $\mu_\epsilon^{-1/2} u_\epsilon(x) \leq C|x - x_\epsilon|^{-1}$ for all $\epsilon > 0$ and $x \in \Omega$. Therefore

$$\int_{B_{x_\epsilon}(\delta)} a_\epsilon u_\epsilon^2 dx = O(\mu_\epsilon) \int_{B_{x_\epsilon}(\delta)} |x|^{-2} dx = O(\delta \mu_\epsilon) \text{ as } \epsilon \rightarrow 0.$$

□

Step 3: We prove Theorem 2 for $n \geq 4$. From the Pohozaev identity (53) we have

$$\begin{aligned} & \mu_\epsilon^{-2} \int_{B_{x_\epsilon}(\delta)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2} \right) u_\epsilon^2 dx - \mu_\epsilon^{-2} \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon}} \frac{(x, x_\epsilon)}{|x|^2} dx \\ &= \mu_\epsilon^{n-4} \int_{\partial B_{x_\epsilon}(\delta)} (x - x_\epsilon, \nu) \left(\frac{|\nabla(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)|^2}{2} + \frac{a}{2} (\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)^2 - \frac{\mu_\epsilon^{2-s_\epsilon}}{2^*(s_\epsilon)} \frac{(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) d\sigma \\ & \quad - \mu_\epsilon^{n-4} \int_{\partial B_{x_\epsilon}(\delta)} \left((x - x_\epsilon, \nabla(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon)) + \frac{n-2}{2} (\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon) \right) \partial_\nu(\mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon) d\sigma \end{aligned} \quad (57)$$

Passing to the limits as $\epsilon \rightarrow 0$ in (57), using (55), (56) and Theorem 6.1, we get Theorem 2 when $n \geq 4$.

Step 4: We now deal with the case of dimension $n = 3$. Recall from the introduction that we write the Green's function G as $G_x(y) = \frac{1}{4\pi|x-y|} + g_x(y)$ for all $x, y \in \Omega$, $x \neq y$, with $g_x \in C^2(\overline{\Omega} \setminus \{x\}) \cap C^{0,\theta}(\Omega)$ for some $0 < \theta < 1$. In particular, when $n = 3$ or $a \equiv 0$, $g_x(x)$ is defined for all $x \in \Omega$. For any $x \in \Omega$, g_x satisfies the equation

$$\Delta g_x + a g_x = -a/(4\pi|x-y|) \text{ in } \Omega \setminus \{x\} \text{ and } g_x(y) = \frac{-1}{\omega_2|x-y|} \text{ on } \partial\Omega.$$

Note that any $x \in \Omega$

$$\lim_{r \rightarrow 0} \sup_{y \in \partial B_x(r)} |y - x| |\nabla g_x(y)| = 0 \quad (58)$$

The proof goes as in Hebey-Robert [13]. We omit it here. From the Pohozaev identity (53), multiplying both the sides by μ_ϵ^{-1} we obtain

$$\begin{aligned} & \int_{B_{x_\epsilon}(\delta)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2} \right) (\mu_\epsilon^{-1/2} u_\epsilon)^2 dx - \frac{s_\epsilon}{2\mu_\epsilon(3-s_\epsilon)} \int_{B_{x_\epsilon}(\delta)} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon}} \frac{(x, x_\epsilon)}{|x|^2} dx = \\ & \int_{\partial B_{x_\epsilon}(\delta)} (x - x_\epsilon, \nu) \left(\frac{|\nabla(\mu_\epsilon^{-1/2} u_\epsilon)|^2}{2} + a \frac{(\mu_\epsilon^{-1/2} u_\epsilon)^2}{2} - \frac{\mu_\epsilon^{2-s_\epsilon}}{2^*(s_\epsilon)} \frac{(\mu_\epsilon^{-1/2} u_\epsilon)^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) d\sigma \\ & \quad - \int_{\partial B_{x_\epsilon}(\delta)} \left((x - x_\epsilon, \nabla(\mu_\epsilon^{-1/2} u_\epsilon)) + \frac{n-2}{2} (\mu_\epsilon^{-1/2} u_\epsilon) \right) \partial_\nu(\mu_\epsilon^{-1/2} u_\epsilon) d\sigma \end{aligned} \quad (59)$$

It follows from Proposition 6.1 that

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \int_{\partial B_{x_\epsilon}(\delta)} (x - x_\epsilon, \nu) \left(\frac{|\nabla(\mu_\epsilon^{-1/2} u_\epsilon)|^2}{2} + a \frac{(\mu_\epsilon^{-1/2} u_\epsilon)^2}{2} - \frac{\mu_\epsilon^{2-s_\epsilon}}{2^*(s_\epsilon)} \frac{(\mu_\epsilon^{-1/2} u_\epsilon)^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) d\sigma \\
& - \lim_{\epsilon \rightarrow 0} \int_{\partial B_{x_\epsilon}(\delta)} \left((x - x_\epsilon, \nabla(\mu_\epsilon^{-1/2} u_\epsilon)) + \frac{n-2}{2} (\mu_\epsilon^{-1/2} u_\epsilon) \right) \partial_\nu(\mu_\epsilon^{-1/2} u_\epsilon) d\sigma \\
& = b_3^2 \int_{\partial B_{x_0}(\delta)} \delta \frac{|\nabla G_{x_0}|^2}{2} + \frac{\delta a}{2} (G_{x_0})^2 - \frac{(x - x_0, \nabla G_{x_0})^2}{\delta} - \frac{n-2}{2} \frac{(x - x_0, \nabla G_{x_0})}{\delta} G_{x_0} d\sigma
\end{aligned}$$

Using (58), we get that the right-hand-side goes to $\frac{b_3^2}{2} g_{x_0}(x_0)$ as $\delta \rightarrow 0$. Putting this identity, (56) when $n = 3$, and (55) in (59), we get Theorem 2 in the case $n = 3$. The proof is similar when $a \equiv 0$.

7. LOCALIZING THE SINGULARITY: THE BOUNDARY BLOW-UP CASE

This section is devoted to the proof of Theorem 3.

7.1. Convergence to Singular Harmonic Functions. Here, G is still the Green's function of the coercive operator $\Delta + a$ in Ω with Dirichlet boundary conditions. The following result for the asymptotic analysis of the Green's function is in the spirit of Proposition 5 of [19] and Proposition 12 of [8].

Theorem 6 ([8, 19]). *Let $(x_\epsilon)_{\epsilon>0} \in \Omega$ and let $(r_\epsilon)_{\epsilon>0} \in (0, +\infty)$ be such that $\lim_{\epsilon \rightarrow 0} r_\epsilon = 0$.*

- (1) *Assume that $\lim_{\epsilon \rightarrow 0} \frac{d(x_\epsilon, \partial\Omega)}{r_\epsilon} = +\infty$. Then for all $x, y \in \mathbb{R}^n$, $x \neq y$, we have that*

$$\lim_{\epsilon \rightarrow 0} r_\epsilon^{n-2} G(x_\epsilon + r_\epsilon x, x_\epsilon + r_\epsilon y) = \frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}$$

where ω_{n-1} is the area of the $(n-1)$ -sphere. Moreover for a fixed $x \in \mathbb{R}^n$, this convergence holds uniformly in $C_{loc}^2(\mathbb{R}^n \setminus \{x\})$.

- (2) *Assume that $\lim_{\epsilon \rightarrow 0} \frac{d(x_\epsilon, \partial\Omega)}{r_\epsilon} = \rho \in [0, +\infty)$. Then $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in \partial\Omega$. Let \mathcal{T} be a parametrisation of the boundary $\partial\Omega$ as in (8) around the point $p = x_0$. We write $\mathcal{T}^{-1}(x_\epsilon) = ((x_\epsilon)_1, x'_\epsilon)$. Then for all $x, y \in \mathbb{R}^n \cap \{x_1 \leq 0\}$, $x \neq y$, we have that*

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} r_\epsilon^{n-2} G(\mathcal{T}((0, x'_\epsilon) + r_\epsilon x), \mathcal{T}((0, x'_\epsilon) + r_\epsilon y)) \\
& = \frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}} - \frac{1}{(n-2)\omega_{n-1}|\pi(x) - y|^{n-2}}
\end{aligned}$$

where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $\pi((x_1, x')) \mapsto (-x_1, x')$ is the reflection across the plane $\{x : x_1 = 0\}$. Moreover for a fixed $x \in \overline{\mathbb{R}^n_-}$, this convergence holds uniformly in $C_{loc}^2(\overline{\mathbb{R}^n_-} \setminus \{x\})$.

The next proposition shows that the pointwise behaviour of the blowup sequence $(u_\epsilon)_{\epsilon>0}$ is well approximated by bubbles. Note that the following proposition holds

with $x_0 \in \overline{\Omega}$, in the interior or on the boundary. We omit the proof as it goes exactly like the proof of Proposition 13 in [8].

Proposition 7.1. *We set for all $\epsilon > 0$*

$$U_\epsilon(x) = \left(\frac{k_\epsilon}{k_\epsilon^2 + \frac{|x-x_\epsilon|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

Suppose that the sequence $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$, where for each $\epsilon > 0$, u_ϵ satisfies (2) and (3), is a blowup sequence. We let $x_0 := \lim_{\epsilon \rightarrow 0} x_\epsilon$. Let $(y_\epsilon)_{\epsilon>0}$ be a sequence of points in $\overline{\Omega}$. We have

- (1) *If $\lim_{\epsilon \rightarrow 0} y_\epsilon = y_0 \neq x_0$, then $\lim_{\epsilon \rightarrow 0} \mu_\epsilon^{-\frac{n-2}{2}} u_\epsilon(y_\epsilon) = b_n G_{x_0}(y_0)$ (see Proposition 6.1).*

- (2) *If $\lim_{\epsilon \rightarrow 0} y_\epsilon = x_0$ and $\lim_{\epsilon \rightarrow 0} d(x_\epsilon, \partial\Omega) > 0$, then*

$$u(y_\epsilon) = (1 + o(1))U_\epsilon(y_\epsilon) \quad \text{as } \epsilon \rightarrow 0$$

- (3) *If $\lim_{\epsilon \rightarrow 0} y_\epsilon = x_0$ and $\lim_{\epsilon \rightarrow 0} d(x_\epsilon, \partial\Omega) = 0$, then*

$$u(y_\epsilon) = (1 + o(1)) \left(U_\epsilon(y_\epsilon) - \tilde{U}_\epsilon(y_\epsilon) \right) \quad \text{as } \epsilon \rightarrow 0$$

where for $\epsilon > 0$

$$\tilde{U}_\epsilon(x) = \left(\frac{k_\epsilon}{k_\epsilon^2 + \frac{|x-\pi_\mathcal{T}(x_\epsilon)|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

where $\pi_\mathcal{T} = \mathcal{T} \circ \pi \circ \mathcal{T}^{-1}$. Here, \mathcal{T} and π are as in Theorem 6.

Using Proposition 7.1, we derive the following when the sequence of blowup points converge to a point on the boundary

Proposition 7.2. *Let $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$ be such that for each $\epsilon > 0$, u_ϵ satisfies (2) and (3). We assume that $u_\epsilon \rightarrow 0$ weakly in $H_{1,0}^2(\Omega)$ as $\epsilon \rightarrow 0$. We let $x_0 := \lim_{\epsilon \rightarrow 0} x_\epsilon$. Let $r_\epsilon = d(x_\epsilon, \partial\Omega)$. We assume that $\lim_{\epsilon \rightarrow 0} r_\epsilon = 0$. Therefore, $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in \partial\Omega$. Let \mathcal{T} be a parametrisation of the boundary $\partial\Omega$ as in (8) around the point $p = x_0$. We write $\mathcal{T}^{-1}(x_\epsilon) = ((x_\epsilon)_1, x'_\epsilon)$. For $\epsilon > 0$, let*

$$\tilde{v}_\epsilon(x) := \frac{r_\epsilon^{n-2}}{\mu_\epsilon^{\frac{n-2}{2}}} u_\epsilon \circ \mathcal{T}((0, x'_\epsilon) + r_\epsilon x) \quad \text{for } x \in \frac{U - (0, x'_\epsilon)}{r_\epsilon} \cap \{x_1 \leq 0\}$$

Then

$$\lim_{\epsilon \rightarrow 0} \tilde{v}_\epsilon(x) = (n(n-2))^{\frac{n-2}{2}} \left(\frac{1}{|x - \theta_0|^{n-2}} - \frac{1}{|x - \pi(\theta_0)|^{n-2}} \right) \quad \text{in } C_{loc}^1(\overline{\mathbb{R}_-^n} \setminus \{\theta_0\})$$

where

$$(60) \quad \theta_0 = \lim_{\epsilon \rightarrow 0} \theta_\epsilon, \quad \theta_\epsilon = \left(\frac{(x_\epsilon)_1}{r_\epsilon}, 0 \right) \in \mathbb{R}_-^n$$

and $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $\pi((x_1, x')) \mapsto (-x_1, x')$ is the reflection across the plane $\{x : x_1 = 0\}$.

Proof. Since $D_0\mathcal{T} = \mathbb{I}_{\mathbb{R}^n}$ we have: $d(x_\epsilon, \partial\Omega) = (1 + o(1))|(x_\epsilon)_1|$. Let θ_ϵ be as in (60). Then we have that $\theta_0 = \lim_{\epsilon \rightarrow 0} \theta_\epsilon = (-1, 0) \in \mathbb{R}_-^n$ and $\pi(\theta_0) = (1, 0) \in \mathbb{R}_+^n$. We fix $R > 0$. \tilde{v}_ϵ is defined in $B_0(R) \cap \{x_1 \leq 0\}$ for $\epsilon > 0$ small. It follows from the strong upper bounds obtained in Theorem 1 that there exists a constant $C > 0$ such that for $\epsilon > 0$ small we have

$$0 \leq \tilde{v}_\epsilon(x) \leq C \left(\frac{r_\epsilon^2}{|\mathcal{T}((0, x'_\epsilon) + r_\epsilon x) - x_\epsilon|^2} \right)^{\frac{n-2}{2}} \quad \text{for } x \in B_0(R) \cap \{x_1 < 0\}$$

For any $x \in B_0(R) \cap \{x_1 \leq 0\}$ we get from Proposition 7.1 that as $\epsilon \rightarrow 0$

$$(61) \quad \tilde{v}_\epsilon(x) = (1 + o(1)) \left(\frac{k_\epsilon}{\mu_\epsilon} \right)^{\frac{n-2}{2}} \left(\left(\frac{1}{\left(\frac{k_\epsilon}{r_\epsilon} \right)^2 + \frac{|\mathcal{T}((0, x'_\epsilon) + r_\epsilon x) - x_\epsilon|^2}{n(n-2)r_\epsilon^2}} \right)^{\frac{n-2}{2}} - \left(\frac{1}{\left(\frac{k_\epsilon}{r_\epsilon} \right)^2 + \frac{|\mathcal{T}((0, x'_\epsilon) + r_\epsilon x) - \pi_{\mathcal{T}^{-1}}(x_\epsilon)|^2}{n(n-2)r_\epsilon^2}} \right)^{\frac{n-2}{2}} \right)$$

From the properties of the boundary map \mathcal{T} , one then gets

$$(62) \quad \lim_{\epsilon \rightarrow 0} \tilde{v}_\epsilon(x) = \frac{(n(n-2))^{\frac{n-2}{2}}}{|x - (1, 0)|^{n-2}} - \frac{(n(n-2))^{\frac{n-2}{2}}}{|x + (1, 0)|^{n-2}} \quad \text{for } x \in (B_0(R) \setminus \{(1, 0)\}) \cap \{x_1 \leq 0\}$$

For $i, j = 1, \dots, n$, we let $(\tilde{g}_\epsilon)_{ij}(x) = (\partial_i \mathcal{T}((0, x'_\epsilon) + r_\epsilon x), \partial_j \mathcal{T}((0, x'_\epsilon) + r_\epsilon x))$, the induced metric on the domain $B_0(R) \cap \{x_1 < 0\}$, and let Δ_g denote the Laplace-Beltrami operator with respect to the metric g . From eqn (2) it follows that given any $R > 0$, \tilde{v}_ϵ weakly satisfies the following equation for $\epsilon > 0$ sufficiently small

$$(63) \quad \begin{cases} \Delta_{\tilde{g}_\epsilon} \tilde{v}_\epsilon + r_\epsilon^2 (a \circ \mathcal{T}((0, x'_\epsilon) + r_\epsilon x)) \tilde{v}_\epsilon = \left(\frac{\mu_\epsilon}{r_\epsilon} \right)^{2-s_\epsilon} \frac{\tilde{v}_\epsilon^{2^*(s_\epsilon)-1}}{\left| \frac{\mathcal{T}((0, x'_\epsilon) + r_\epsilon x)}{r_\epsilon} \right|^{s_\epsilon}} & \text{in } B_0(R) \cap \{x_1 < 0\} \\ \tilde{v}_\epsilon = 0 & \text{on } B_0(R) \cap \{x_1 = 0\} \end{cases}$$

Arguing as in Step 1.2 of the proof of Lemma 1, we get that the convergence of \tilde{v}_ϵ holds in $C_{loc}^1(\overline{\mathbb{R}_-^n} \setminus \{\theta_0\})$. This completes the proof of Proposition 7.2. \square

7.2. Estimates on the blow up rates: The Boundary Case. Suppose that the sequence of blow up points $(x_\epsilon)_{\epsilon > 0}$ converges to a point on the boundary, i.e suppose $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in \partial\Omega$. We let

$$(64) \quad r_\epsilon = d(x_\epsilon, \partial\Omega)$$

Then $\lim_{\epsilon \rightarrow 0} r_\epsilon = 0$ and from (17), we have as $\epsilon \rightarrow 0$: $\mu_\epsilon = o(r_\epsilon)$ and $k_\epsilon = o(r_\epsilon)$. We apply the Pohozaev identity for the Hardy Sobolev equation (53) to the domain $B_{x_\epsilon}(r_\epsilon/2)$. Note that since $\frac{d(x_\epsilon, \partial\Omega)}{r_\epsilon} = 1$ for all $\epsilon > 0$, so $\overline{B_{x_\epsilon}(r_\epsilon/2)} \subset \subset \Omega$ for $\epsilon > 0$ small. The Pohozaev identity (53) gives us

$$(65) \quad \begin{aligned} & \int_{B_{x_\epsilon}(r_\epsilon/2)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2} \right) u_\epsilon^2 dx - \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(r_\epsilon/2)} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon}} \frac{1}{|x|^2} dx = \\ & \int_{\partial B_{x_\epsilon}(r_\epsilon/2)} (x - x_\epsilon, \nu) \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) - \left((x - x_\epsilon, \nabla u_\epsilon) + \frac{n-2}{2} u_\epsilon \right) \partial_\nu u_\epsilon d\sigma \end{aligned}$$

for all $\epsilon > 0$ small. We now estimate each of the terms in the integral above. Theorem 3 will be a consequence of the following theorem:

Theorem 7. *Let Ω , a , $(s_\epsilon)_{\epsilon>0}$, $(u_\epsilon)_{\epsilon>0} \in H_{1,0}^2(\Omega)$ as in Theorem 3. Assume that (64) holds and $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in \partial\Omega$. Then*

(1) *If $n = 3$ or $a \equiv 0$, then as $\epsilon \rightarrow 0$*

$$(66) \quad \lim_{\epsilon \rightarrow 0} \frac{s_\epsilon r_\epsilon^{n-2}}{\mu_\epsilon^{n-2}} = \frac{n^{n-1}(n-2)^{n-1}K(n,0)^{n/2}\omega_{n-1}}{2^{n-2}}.$$

Moreover, $d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_\epsilon|$ as $\epsilon \rightarrow 0$. In particular $x_0 = 0$.

(2) *If $n = 4$. Then as $\epsilon \rightarrow 0$*

$$(67) \quad \begin{aligned} & \frac{s_\epsilon}{4} (K(4,0)^{-2} + o(1)) - \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^2 (32\omega_3 + o(1)) = \mu_\epsilon^2 \log\left(\frac{r_\epsilon}{\mu_\epsilon}\right) [d_4 a(x_0) + o(1)] \\ & \text{and} \\ & s_\epsilon \left(1 - \left(\frac{r_\epsilon}{|x_\epsilon|}\right)^2 + o(1)\right) = \mu_\epsilon^2 \log\left(\frac{r_\epsilon}{\mu_\epsilon}\right) [4d_4 K(4,0)^2 a(x_0) + o(1)] \end{aligned}$$

(3) *If $n \geq 5$. Then as $\epsilon \rightarrow 0$*

$$(68) \quad \begin{aligned} & \frac{s_\epsilon(n-2)}{2n} (K(n,0)^{-n/2} + o(1)) - \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{n-2} \left(\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}} + o(1)\right) = \mu_\epsilon^2 [d_n a(x_0) + o(1)] \\ & \text{and} \\ & s_\epsilon \left(1 - \left(\frac{r_\epsilon}{|x_\epsilon|}\right)^2 + o(1)\right) = \mu_\epsilon^2 \left[\frac{2n}{n-2} d_n K(n,0)^2 a(x_0) + o(1)\right] \end{aligned}$$

where d_n is as in (4) for $n \geq 5$ and $d_4 = 64\omega_3$.

Proof. For convenience we define

$$F_\epsilon = (x - x_\epsilon, \nu) \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) - \left((x - x_\epsilon, \nabla u_\epsilon) + \frac{n-2}{2} u_\epsilon \right) \partial_\nu u_\epsilon$$

Step 1: We claim that

$$(69) \quad \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-n} \int_{\partial B_{x_\epsilon}(r_\epsilon/2)} F_\epsilon \, d\sigma = -\frac{n^{n-2}(n-2)^n\omega_{n-1}}{2^{n-1}} + o(1) \quad \text{as } \epsilon \rightarrow 0$$

Proof. We define

$$(70) \quad \hat{v}_\epsilon(x) := \frac{r_\epsilon^{n-2}}{\mu_\epsilon^{\frac{n-2}{2}}} u_\epsilon(x_\epsilon + r_\epsilon x) \text{ for } x \in B_0(1).$$

Since $d(x_\epsilon, \partial\Omega) = r_\epsilon$, this is well-defined. Moreover, Proposition 7.2 yields

$$(71) \quad \lim_{\epsilon \rightarrow 0} \hat{v}_\epsilon = \hat{v}(x) = \frac{(n(n-2))^{\frac{n-2}{2}}}{|x|^{n-2}} - \frac{(n(n-2))^{\frac{n-2}{2}}}{|x - (2,0)|^{n-2}} \text{ in } C_{loc}^1(B_0(1)).$$

With the change of variable $x \mapsto x_\epsilon + r_\epsilon z$ we obtain

$$\begin{aligned} \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-n} \int_{\partial B_{x_\epsilon}(r_\epsilon/2)} F_\epsilon \, d\sigma &= \int_{\partial B_0(1/2)} (z, \nu) \frac{|\nabla \hat{v}_\epsilon|^2}{2} \, d\sigma \\ &+ \int_{\partial B_0(1/2)} (z, \nu) r_\epsilon^2 a(x_\epsilon + r_\epsilon z) \frac{\hat{v}_\epsilon^2}{2} \, d\sigma - \int_{\partial B_0(1/2)} \frac{(z, \nu)}{2^*(s_\epsilon)} \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-s_\epsilon} \frac{\hat{v}_\epsilon^{2^*(s_\epsilon)}}{\left|\frac{x_\epsilon + r_\epsilon z}{r_\epsilon}\right|^{s_\epsilon}} \, d\sigma \\ &- \int_{\partial B_0(1/2)} \left((z, \nabla \hat{v}_\epsilon) + \frac{n-2}{2} \hat{v}_\epsilon \right) \partial_\nu \hat{v}_\epsilon \, d\sigma \end{aligned}$$

Passing to limit as $\epsilon \rightarrow 0$ in (72) and using (71), we get

$$(72) \quad \left(\frac{\mu_\epsilon}{r_\epsilon}\right)^{2-n} \int_{\partial B_{x_\epsilon}(r_\epsilon/2)} F_\epsilon \, d\sigma = A(1/2) + o(1) \text{ as } \epsilon \rightarrow 0$$

where

$$A(\delta) := \int_{\partial B_0(\delta)} \left((z, \nu) \frac{|\nabla \hat{v}|^2}{2} - \left((z, \nabla \hat{v}) + \frac{n-2}{2} \hat{v} \right) \partial_\nu \hat{v} \right) \, d\sigma.$$

Let $0 < \delta < 1/2$. Since $\Delta \hat{v} = 0$ in $B_0(1/2) \setminus B_0(\delta)$, applying the Pohozaev identity (52), we see that $A(\delta) = A(1/2)$ for all $0 < \delta < 1/2$. We write

$$(73) \quad \hat{v}(x) = \frac{(n(n-2))^{\frac{n-2}{2}}}{|x|^{n-2}} + h(x) \quad \text{for } x \in B_0(1) \setminus \{0\}$$

where $h(x) = -\frac{(n(n-2))^{\frac{n-2}{2}}}{|x+(2,0)|^{n-2}}$. With the explicit expression of v we obtain

$$\lim_{\delta \rightarrow 0} A(\delta) = -\frac{n^{n-2}(n-2)^n \omega_{n-1}}{2^{n-1}}$$

Since A is constant, this latest limit and (72) yield (69). This completes Step 1. \square

Step 2: Proceeding similarly as in (55) we obtain

$$\int_{B_{x_\epsilon}(r_\epsilon/2)} \frac{u_\epsilon^{2^*(s_\epsilon)}(x, x_\epsilon)}{|x|^{s_\epsilon} |x|^2} \, dx = \left(\frac{1}{K(n, 0)} \right)^{\frac{2^*}{2^*-2}} + o(1) \quad \text{as } \epsilon \rightarrow 0$$

Step 3: Arguing as in the proof of (56), we get that

$$\int_{B_{x_\epsilon}(r_\epsilon/2)} \left(a + \frac{(x - x_\epsilon, \nabla a)}{2} \right) u_\epsilon^2 \, dx = \begin{cases} O(\mu_\epsilon) & \text{for } n = 3 \text{ or } a \equiv 0, \\ \mu_\epsilon^2 \log\left(\frac{r_\epsilon}{k_\epsilon}\right) [64\omega_3 a(x_0) + o(1)] & \text{for } n = 4, \\ \mu_\epsilon^2 [d_n a(x_0) + o(1)] & \text{for } n \geq 5. \end{cases}$$

where d_n is as in (4). Combining Steps 1 to 3 in the Pohozaev identity (65) yields

(66), (67) and (68).

To get extra informations, we differentiate the Pohozaev identity (53) with respect to the j^{th} variable $(x_\epsilon)_j$ and get

$$(74) \quad \int_{B_{x_\epsilon}(r_\epsilon/2)} \frac{\partial_j a}{2} u_\epsilon^2 dx + \frac{s_\epsilon(n-2)}{2(n-s_\epsilon)} \int_{B_{x_\epsilon}(r_\epsilon/2)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{x_j}{|x|^2} dx = \int_{\partial B_{x_\epsilon}(r_\epsilon/2)} \left(\nu_j \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) - \partial_j u_\epsilon \partial_\nu u_\epsilon \right) d\sigma$$

Step 4: We claim that

$$(75) \quad \frac{\mu_\epsilon^{2-n}}{r_\epsilon^{1-n}} \int_{\partial B_{x_\epsilon}(r_\epsilon/2)} \left(\nu_1 \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) - \partial_1 u_\epsilon \partial_\nu u_\epsilon \right) d\sigma = -\frac{n^{n-2}(n-2)^n \omega_{n-1}}{2^{n-1}} + o(1)$$

Proof. As Step 1 above, using Proposition 7.2 we have as $\epsilon \rightarrow 0$

$$\begin{aligned} & \frac{\mu_\epsilon^{2-n}}{r_\epsilon^{1-n}} \int_{\partial B_{x_\epsilon}(r_\epsilon/2)} \left(\nu_j \left(\frac{|\nabla u_\epsilon|^2}{2} + \frac{a u_\epsilon^2}{2} - \frac{1}{2^*(s_\epsilon)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \right) - \partial_j u_\epsilon \partial_\nu u_\epsilon \right) d\sigma \\ &= \int_{\partial B_0(1/2)} \left(\nu_j \frac{|\nabla \hat{v}|^2}{2} - \partial_j \hat{v} \partial_\nu \hat{v} \right) d\sigma + o(1) \end{aligned}$$

where \hat{v}_ϵ and \hat{v} are as in Step 1 above. Arguing as in Step 1 above, we get that

$$(76) \quad \int_{\partial B_0(1/2)} \left(\nu_j \frac{|\nabla \hat{v}|^2}{2} - \partial_j \hat{v} \partial_\nu \hat{v} \right) d\sigma = \omega_{n-1}(n-2)(n(n-2))^{\frac{n-2}{2}} \partial_j h(0),$$

where h is as in (73). For $j = 1$, taking the explicit expression of h yields Step 4. \square

Step 5: Arguing as in Step 2 we have

$$(77) \quad \int_{B_{x_\epsilon}(r_\epsilon/2)} \frac{u_\epsilon^{2^*(s_\epsilon)}}{|x|^{s_\epsilon}} \frac{x_1}{|x|^2} dx = \frac{(x_\epsilon)_1}{|x_\epsilon|^2} \left(\frac{1}{K(n,0)} \right)^{\frac{2^*}{2^*-2}} (1 + o(1)) \quad \text{as } \epsilon \rightarrow 0$$

Similarly, as in Step 3, for every $1 \leq j \leq n$ we have as $\epsilon \rightarrow 0$

$$(78) \quad \int_{B_{x_\epsilon}(r_\epsilon/2)} \partial_j a(x) u_\epsilon^2(x) dx = \begin{cases} O(\mu_\epsilon) & \text{for } n = 3, \\ O\left(\mu_\epsilon^2 \log\left(\frac{r_\epsilon}{k_\epsilon}\right)\right) & \text{for } n = 4, \\ O(\mu_\epsilon^2) & \text{for } n \geq 5. \end{cases}$$

Using the Pohozaev identity (74), (68) and these estimates, noting that $r_\epsilon = d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_{\epsilon,1}|$, we then obtain that $d(x_\epsilon, \partial\Omega) = (1 + o(1))|x_\epsilon|$ as $\epsilon \rightarrow 0$ when $n = 3$ or $a \equiv 0$. When $n = 4$, then as $\epsilon \rightarrow 0$

$$\frac{s_\epsilon}{4} \frac{(x_\epsilon)_1}{|x_\epsilon|^2} (K(4,0)^{-2} + o(1)) + \frac{\mu_\epsilon^2}{r_\epsilon^3} (32\omega_3 + o(1)) = O\left(\mu_\epsilon^2 \log\left(\frac{r_\epsilon}{\mu_\epsilon}\right)\right).$$

Finally, when $n \geq 5$, we get as $\epsilon \rightarrow 0$

$$\begin{aligned} & \frac{s_\epsilon(n-2)}{2n} \frac{(x_\epsilon)_1}{|x_\epsilon|^2} \left(K(n, 0)^{-n/2} + o(1) \right) + r_\epsilon^{-1} \left(\frac{\mu_\epsilon}{r_\epsilon} \right)^{n-2} \left(\frac{n^{n-2}(n-2)^n \omega_{n-1}}{2^{n-1}} + o(1) \right) \\ & = O(\mu_\epsilon^2) \end{aligned}$$

Plugging together these estimates and (67) and (68), we get Theorem 7. \square

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